ADVANCED LABORATORY PRACTICE IN ELECTRICITY AND MAGNETISM

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PREFACE TO THE THIRD EDITION

In this, the third edition, a number of important changes have been made. Experiment (3) has been revised and Experiment (48) has been changed in order to emphasize the difference between the root mean square and the average value of an alternating current.

In addition two new experiments have been added, *viz.*, the determination of the electronic charge and the determination of the thermionic work function of a metal. Both of these experiments have been in use in this laboratory for a number of years and have proved very satisfactory.

The chapter on electron tubes has been partially rewritten in order to give the student some familiarity with the recent electron theory of metals, in particular as it applies to thermionic emission.

The writer wishes to express his thanks to Dr. L. J. Haworth for his cooperation.

H. B. WAHLIN.

UNIVERSITY OF WISCONSIN,
MADISON, WIS.,
September, 1935.

PREFACE TO THE FIRST EDITION

In preparing this book, the author has had in mind particularly the needs of those students who have at their disposal only one year to devote to the study of electricity and magnetism in addition to the work covered in an elementary course in general physics. It has been his aim to include, in addition to the usual work in electrical measurements, a sufficient study of the discharge of electricity through gases, radio activity, and thermionics to enable those who cannot pursue special courses to gain an idea of the fundamentals of these newer branches.

The subject matter covers the work given to third year students in electrical engineering at the University of Wisconsin. Following the elementary work of the first nine chapters, a number of the complex bridge methods for precise measurements of inductance and capacitance are discussed, together with descriptions of the various sources of alternating currents which have been developed in recent years for energizing bridge circuits. A discussion of the more modern instruments for detecting the balance condition of bridges, together with their individual merits, has been included. This is preceded by an elementary study of "transients," in which the fundamental phenomena of reactance, necessary for an understanding of bridge methods, are set forth.

The electron tube, because of the multiplicity of its uses, finds many applications, not only in the art of radio communication, but also in engineering practice and in the general research laboratory. Considerable space has been devoted to this device, as well as to the fundamentals of the electron theory and the passage of electricity through gases.

The author is a firm believer in the laboratory method of instruction, and each exercise is preceded by a discussion of the theories involved sufficient to enable the student to understand clearly the relation of each experiment to the general field in which it lies. It is believed that with the material given in the text and the references to standard works, which have been included, the student can pursue the subject without the aid of

formal lectures, although at the present time the writer is devoting one hour per week to a lecture-conference. Experience shows that the average student performs fourteen of these exercises per semester and the topics herewith presented accordingly permit of some little choice.

In selecting material, advantage has been taken not only of the original sources, but also of the standard texts in the special fields represented. Being a collection of laboratory exercises, this book makes no claim to originality of the subject matter included, and the author hereby acknowledges his indebtedness to the many books and special articles referred to in the footnotes throughout the text. He is indebted, also, to the Leeds-Northrup Company, J. G. Biddle Company, Queen & Company, General Radio Company, Tinsley & Company, and other manufacturers of electrical apparatus for supplying the cuts which have been used. In particular, he wishes to express his gratitude to Dr. H. B. Wahlin and L. L. Nettleton, instructors in physics at the University of Wisconsin, who have read the entire manuscript and made many valuable suggestions during its preparation.

E. M. Terry.

University of Wisconsin,
Madison, Wis.,
June, 1922.

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ADVANCED LABORATORY PRACTICE

IN

ELECTRICITY AND MAGNETISM

CHAPTER I

GENERAL DIRECTIONS—ELECTRICAL UNITS

1. Preparation.—If one is to make the best use of his time in the laboratory, he must understand thoroughly what is to be done and then proceed in a systematic manner to do it. This can be accomplished only when preparation for the task has been made before taking up the experimental work. Assignments accordingly will be made one week in advance, and the student is expected to enter the laboratory with the following preparation:

1. An understanding of the theory of the experiment.
2. A knowledge of the working principles of the instruments to be used.
3. A schedule according to which the data are to be taken.

In order to facilitate the work of the first few periods, the following general directions should be carefully read:

2. Connections.—A large portion of the trouble in performing electrical measurements arises from imperfect connections. All instruments, to which wires are to be attached, are provided with binding posts. To secure good contact, remove the insulation about an inch from the end of the wire, scrape it clean, wrap it two-thirds around the binding post, and then screw down the nut. If the wire is too short to reach between the points desired, join two or more wires with connectors, having first scraped the ends clean. Never join wires by twisting their ends together, as connections of this sort, unless soldered, are entirely unreliable. Do not coil wires about a rod or a pencil, since then they cannot be used again. Cut wires to the proper length, thus avoiding a complicated tangle difficult to trace, which, through leaks,

furnishes a source of constant trouble. Never allow one wire to rest upon another, even though both are covered with insulation.

Before attempting a set-up, make a rough sketch of connections, arranging the apparatus in a compact and orderly manner. This will be of great service later in checking connections and locating faults. In many cases, especially in complicated net works, a little forethought in the arrangement will save much time and inconvenience in the performance of the test. Always make the connection with the source of current supply last, *having first assured yourself as to the correctness of the connections* by comparison with the sketch, or by consultation with your instructor. As a further precaution, close the main switch at first only an instant, opening it at once to see if there are any indications of a short circuit. This is especially important where the source is a dynamo or a storage battery.

3. Keys and Switches.—Always open and close a switch quickly, to avoid burning it at the point of contact. If the

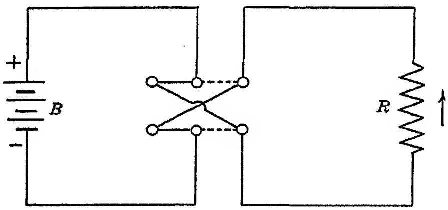

FIG. 1.—Reversing switch.

circuit includes mercury cups and connecting links, it should be broken by means of a knife switch, not by removing the link, as mercury is especially likely to arc. Ordinary contact keys should be used only where a small current is to be carried, and where variations in the resistance of the circuit introduce no serious error; as, for example, the galvanometer circuit of a Wheatstone bridge.

The device generally employed for reversing the current through any portion of a circuit is the Pohl's commutator, which consists of a double pole double throw switch with two cross wires, as shown in Fig. 1. It will be seen that when the switch is closed, as shown by the heavy lines, the current through R flows upwards, but is reversed when the switch is thrown towards the right. Since the cross wires in such a commutator are frequently

placed under the block, a double pole double throw switch should be examined carefully before it is connected in circuit, as the cross wires of the commutator may produce short circuits, resulting in serious injury to the apparatus.

4. Rheostats.—A rheostat is a variable resistance capable of carrying considerable current. It is used primarily as a controlling device and its value, in general, need not be accurately known. When connected in series with a source of electrical

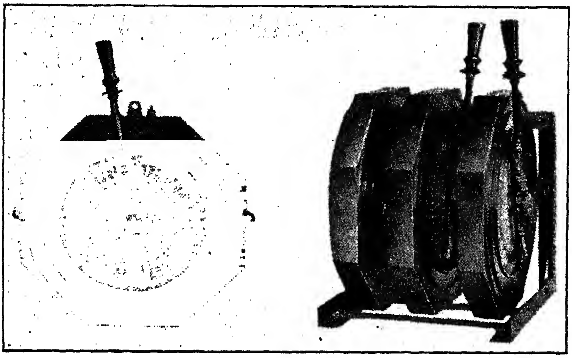

Fig. 2.—Rheostat with fixed steps.

power, the current supplied to any circuit may be varied and brought to any desired value, within certain definite limits, by changing the resistance of the rheostat. Since the energy consumed by a rheostat always appears in the form of heat, the current carrying capacity for a given resistance depends upon the provision made for dissipating heat either by conduction, convection, or radiation.

Many different forms of rheostats are in use and only a few of the more common types will be mentioned here. Figure 2 illustrates one that is frequently used for controlling relatively large currents. It consists of a number of copper lugs between which are connected units of high resistance metal in the form of a thin ribbon to give as much heat radiating surface as possible. They are bent back and forth in a zig-zag shape and embedded in sand. With this arrangement the resistance varies by steps. By connecting two in series, one having large and the other small steps, a fairly smooth variation in current may be obtained.

Another common form of rheostat is shown in Fig. 3. A high resistance wire is wound on an insulating tube. Binding posts are connected to each end of the wire, and as the sliding contact is moved along, the resistance between it and one end of the wire changes from zero to a maximum. Wires of various sizes are frequently wound on the same tube thus giving two or more

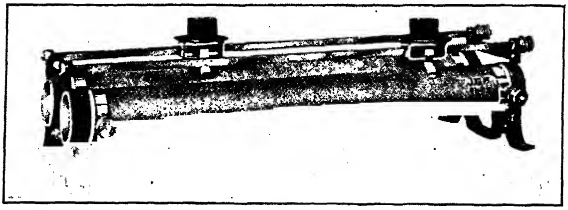

FIG. 3.—Tube rheostat.

ranges for one instrument. For carrying large currents, the ends of the tube are closed and cooling water is passed through. Such rheostats are generally wound with a ribbon to improve the thermal contact between the wire and tube. The figure shows a high resistance instrument, which may also be used as a potential divider. If the E.M.F. to be divided is connected across the end binding posts, any desired fraction of this voltage may be

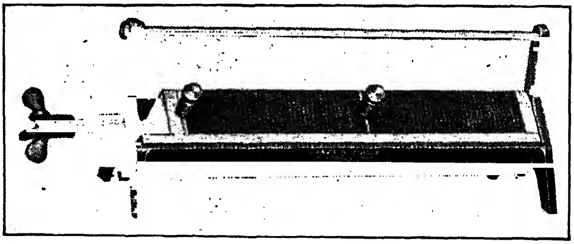

FIG. 4.—Carbon compression rheostat.

obtained by "picking off" between one end and the slider, and moving the latter back and forth.

Another device for controlling current makes use of the fact that the resistance between two carbon surfaces varies with the pressure. An instrument of this sort is shown in Fig. 4. It consists of series of rectangular carbon plates placed in a trough and arranged in such a way that they can be subjected to variable pressures by means of an adjusting screw. Rheostats of this

type are useful where low voltage currents are to be controlled. They have the disadvantage of requiring frequent readjustment since the tension changes with variations of the temperature of both the carbon plates and metal parts.

5. Switch Board.—A switch board is a necessary adjunct to any electrical laboratory and is used to distribute electrical power of different types and voltage to the various working circuits of the laboratory, and to connect the different circuits with one another. It consists of a panel of insulating material, usually marble, on which is mounted a series of pairs of sockets. The various laboratory and power circuits are joined to these sockets on the back of the board and connections between them are made at the front by means of flexible connectors, often called "jumpers," to the ends of which are attached plugs which fit snugly into the sockets. Power circuits are distinguished by the word "Volts." With the exception of the A.C. circuits, each terminal is labeled plus or minus. If, for example, 10 volts are desired on circuit 91, connect the positive of a 10 volt set with the positive of 91, and similarly for the negative, when the polarity at the laboratory end will be found as indicated. If some voltage is desired, e.g., 16 volts, for which there is no separate set, connect two or more sets in series, joining plus to minus as though connecting cells on a table; then, considering the group as a single set, connect to the laboratory terminals as above. If a current larger than the normal rated capacity of the storage battery is desired, use the dynamo or connect in parallel two or more sets of equal voltage. To do the latter, join all the positive terminals, similarly the negatives, and then connect to the laboratory terminals as above. Before making switch board connections, be sure that the circuit switch in the laboratory is open. Connect the "dead" ends first, and, before pushing in the final plug closing the circuit, tap it cautiously against the socket, quickly withdrawing it. If a spark is seen, some error in connection has been made which must be located before the circuit is closed. Never connect in parallel on a battery with some one else without first obtaining his permission.

6. Care of Apparatus.—Electrical apparatus is delicate and expensive, and it is necessary to proceed with the utmost caution. If an instrument is provided with a shunt, use the smallest resistance first; or, if protected by a series resistance, use the largest value first, decreasing it until the desired value has been

reached. If an instrument fails to work, do not replace it in the case and get another, but report at once to the instructor. Resistance boxes are most frequently injured by carrying too large currents. Before closing the main switch, look over the connections and make a rough calculation of the current that will flow in each box. In no case should the power consumed by a single coil, given by I^2R, be more than four watts. Plugs should be seated by a gentle pressure, accompanied by a twisting motion, heavy pressure being unnecessary.

Never move galvanometers from one place to another without first making sure that the weight of the moving system has been removed from the suspension by means of the arrestment which is always provided. Standard cells should never be tipped up for purposes of inspection or otherwise, and should not be used as a source of current, but merely for balancing potentials; and even here, a large series resistance should at first be included and cut out as a balance is approached. Ammeters are instruments for measuring the total current flowing, and should be connected in series with the circuit, analogous to a water meter. They are most frequently injured by the passage of too large currents. If the arrangement of the apparatus is not such that the current can be calculated approximately before the circuit is closed, a sufficiently large rheostat should be included and cautiously cut out, the instrument being watched in the meantime. Voltmeters are electrical pressure gauges, indicating the difference of potential between the points to which their terminals are attached, and are accordingly connected in parallel with the circuit. Most voltmeters are provided with two scales; and in such cases, one should use the larger first, transferring to the smaller one if the voltage is found to be less than the lower full scale reading. Before leaving the laboratory, return all apparatus to its proper place in the cases. Wires less than a foot long should be thrown in the waste box, and the others returned to their hooks in the wire cabinet. Switch board connectors should be pulled and returned to the proper hooks. Leave the laboratory as tidy as you found it.

7. Notebooks.—All data, as they are taken during an experiment, should be recorded in tabular form in a rough notebook with bound leaves. Ascertain from the instructor specific directions regarding the form in which the final report is to be made, and in its preparation observe the following outline:

1. Give name of experiment and references.
2. Enumerate apparatus used, giving number of each piece.
3. Make a sketch (not a picture) including all instruments, resistance boxes, switches, etc., which will show the actual path of the current. (Use a ruler and dividers.)
4. Give the theory of the experiment as fully as possible, deriving all formulae used.
5. Outline briefly the method of procedure, mentioning special precautions to be taken and difficulties to be overcome.
6. Tabulate your data, arranging it in compact form. State the units in which your results are expressed.
7. Plot curves showing your results graphically, using as ordinates the dependent variable. Choose scales such that the curves will cover as nearly as possible the entire sheet, labeling axes and putting the scale along each. Draw in a smooth curve, striking an average between outstanding points.
8. Give a brief discussion of results, including estimates of accuracy and sources of error.
9. Answer all questions asked under special directions at the end of each experiment.

ELECTRICAL AND MAGNETIC UNITS

8. Systems of Units.[1]—There are two distinct systems of units used in the measurement of electrical quantities; the electrostatic and the electromagnetic. In the former, the fundamental unit is determined by means of the repulsion between two similar charges of electricity, while in the latter, it is based upon the repulsion of two similar magnetic poles. Both of these systems may properly be termed "absolute" since all the quantities involved are directly expressible in terms of the fundamental units of length, mass, and time. The ratio between corresponding units of these two systems is some power of the velocity of light. In actual practice, however, neither of these systems is used, since, in general, the quantities therein defined are not of such magnitudes as to be convenient working units. A third system, known as the "practical system," has accordingly been devised, in which all the units are decimal multiples of the corresponding electromagnetic units. The units of this system are

[1] Everett, The C. G. S. System of Units, chap. X, XI.
Electrical Meterman's Handbook, chap. II.

the only ones to which names have been given, and it has been the custom of the international conferences by which they have been defined, to honor scientists, famous in the fields in which the units lie, by giving to them their names. Electrical quantities, expressed in the electrostatic and electromagnetic systems, are designated by the letters E.S.U. and E.M.U., respectively.

FUNDAMENTAL ELECTRICAL UNITS

9. Magnetic Units. *Magnetic Pole Strength.*—The unit magnetic pole is a pole of such strength that it repels a like pole at a distance of one centimeter, in air, with a force of one dyne.

Magnetic Field Strength.—A magnetic field of unit intensity is a field that acts upon a unit magnetic pole placed in it, with a force of one dyne.

10. Electrostatic Units. *Quantity.*—The electrostatic unit of quantity is of such a magnitude that it repels a like quantity at a distance of one centimeter, in air, with a force of one dyne.

Current.—The electrostatic unit of current exists when an electrostatic unit of quantity flows past any plane in a conductor per second.

Potential Difference.—Unit electrostatic difference of potential exists between two points when the amount of work required to carry an electrostatic unit of quantity from one to the other is one erg.

Resistance.—A conductor possesses the electrostatic unit of resistance if, when carrying the electrostatic unit of current, the difference of potential across its terminals is one electrostatic unit.

Capacitance.—A condenser possesses an electrostatic unit of capacitance if the electrostatic unit of potential difference across its terminals gives to it the electrostatic unit of charge.

Inductance.—A coil possesses an electrostatic unit of inductance if, when the inducing current is changing at the rate of one electrostatic unit per second, the induced electromotive force is one electrostatic unit. This applies both to self and mutual induction.

11. Electromagnetic Units. *Current.*—The electromagnetic unit of current is a current such that, when flowing through an arc of one centimeter length of a circle of one centimeter radius, it produces, at the center, a unit magnetic field.

Quantity.—The electromagnetic unit of quantity is that quan-

tity which passes, per second, any plane of a conductor in which the electromagnetic unit of current is flowing.

Potential Difference.—The electromagnetic unit of potential difference exists between two points when the amount of work required to carry the electromagnetic unit of quantity from one to the other is one erg.

Resistance.—A conductor possesses the electromagnetic unit of resistance, if, when carrying the electromagnetic unit of current, the difference of potential across its terminals is one electromagnetic unit.

Capacitance.—A condenser possesses the electromagnetic unit of capacitance if the electromagnetic unit of potential difference across its terminals gives to it one electromagnetic unit of charge.

Inductance.—A coil possesses an electromagnetic unit of inductance if, when the inducing current varies at the rate of one electromagnetic unit per second, the induced electromotive force is one electromagnetic unit.

12. Practical Units. *Current.*—An ampere is one-tenth of an electromagnetic unit of current.

Quantity.—The coulomb is the quantity of electricity which passes per second any plane of a conductor in which the current is one ampere.

Potential Difference.—The difference of potential between two points is one volt when the amount of work required to carry one coulomb from one to the other is one joule.

Resistance.—A conductor possesses a resistance of one ohm if, when carrying a current of one ampere, the difference of potential across its terminals is one volt.

Capacitance.—A condenser possesses a capacitance of one farad if a difference of potential of one volt across its terminals gives it a charge of one coulomb.

Inductance.—Two coils possess a mutual inductance of one henry if, when the primary current is changing at the rate of one ampere per second, the electromotive force induced in the secondary is one volt.

A coil possesses one henry of self-inductance if, when the current through it is varying at the rate of one ampere per second, the induced counter electromotive force is one volt. One millihenry equals 0.001 henry.

Magnetic Flux.—The total flux in a magnetic circuit is one maxwell when it possesses one magnetic line of induction.

Magnetic Induction.—The induction in a magnetic circuit is one gauss when the flux density is one maxwell per square centimeter.

Magnetomotive Force.—The magnetomotive force of a magnetic circuit is one gilbert if the work required to carry a unit magnetic pole once around the circuit is one erg.

Field Strength.—A magnetic field possesses unit strength if the magnetomotive force is one gilbert per centimeter. (This definition is identical with that previously given for field strength.)

Reluctance.—A magnetic circuit possesses a reluctance of one oersted if a magnetomotive force of one gilbert produces a flux of one maxwell.

13. Legal Definitions of the Practical Units.—At the last International Conference on Electrical Units and Standards, which met in London, in 1908, the following resolutions were adopted, which have served as the basis for legislation in the different countries of the world for fixing the legal definitions of the fundamental electrical units now in force. The full report of this Conference, in which 21 different nations were represented, may be found in The Electrical Review, vol. 63, (1908), page 738.

RESOLUTIONS

I. The Conference agrees that as heretofore the magnitude of the fundamental electric units shall be determined on the electromagnetic system of measurements with reference to the centimeter as the unit of length, the gram as the unit of mass, and the second as the unit of time.

These fundamental units are (1) the ohm, the unit of electric resistance which has the value of 1,000,000,000 in terms of the centimeter and second; (2) the ampere, the unit of electric current which has the value of one-tenth (0.1) in terms of the centimeter, gram, and second; (3) the volt, the unit of electromotive force which has the value of 100,000,000 in terms of the centimeter, the gram, and the second; (4) the watt, the unit of power, which has the value of 10,000,000 in terms of the centimeter, the gram, and the second.

II. As a system of units representing the above and sufficiently near to them to be adopted for the purpose of electrical measurements and as a basis for legislation, the Conference recommends the adoption of the International ohm, the International ampere,

and the International volt, defined according to the following definitions.

III. The ohm is the first primary unit.

IV. The International ohm is defined as the resistance of a specified column of mercury.

V. The International ohm is the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14.4521 grams in mass, of a constant cross-sectional area and of a length of 106.300 cm.

To determine the resistance of a column of mercury in terms of the International ohm, the procedure to be followed shall be that set out in specification I, attached to these resolutions.

VI. The ampere is the second primary unit.

VII. The International ampere is the unvarying electric current which, when passed through a solution of nitrate of silver in water, in accordance with the specification II, attached to these resolutions, deposits silver at the rate of 0.00111800 of a gram per second.

VIII. The International volt is the electrical pressure which, when steadily applied to a conductor whose resistance is one International ohm, will produce a current of one International ampere.

IX. The International watt is the energy expended per second by an unvarying electric current of one International ampere under an electric pressure of one International volt.

The Conference recommends the use of the Weston Normal Cell as a convenient method of measuring both electromotive force and current, and when set up under the conditions specified in schedule *C*, may be taken, provisionally, as having, at a temperature of 20° C., an E.M.F. of 1.0184 volts.

14. The New Value of the Weston Standard Cell.—Since the meeting of the London Conference, a large amount of research has been carried on at the Bureau of Standards at Washington on the Weston Cell and the electrochemical equivalent of silver; and it has been found that the electromotive force of this cell, in terms of the International ohm and International ampere, is, within one part in 10,000,

$$E = 1.0183 \text{ International volts at } 20° \text{ C.},$$

and this value was adopted by the Bureau of Standards Jan. 1, 1911. The formula for the temperature coefficient of the Weston Cell adopted by the London Conference, based on the investiga-

tions of the Bureau of Standards, is as follows:

$$E_t = E_{20} - .0000406\ (t - 20°) - .00000095\ (t - 20)^2 + .00000001\ (t - 20)^3 \quad (1)$$

15. Ratios of the Electrical Units.—For convenience of comparison the dimensions of the electrostatic and electromagnetic units are given below. The dimensions of the dielectric constant and the permeability are unknown and are inserted in the formula as κ and μ, respectively. All that is known concerning the nature of these quantities is that $\frac{c}{\sqrt{\kappa\mu}}$ equals v, where v is the velocity of light in a medium having a dielectric constant κ and a permeability μ, and c is the number of absolute c.g.s. electrostatic units of quantity of electricity in one electromagnetic unit of quantity of electricity. It is important to note that c is numerically equal to the velocity of light in free space, i.e., 3×10^{10} cms. per second. The last column gives the ratio of the corresponding units in the two systems in terms of c.

Unit	Electro-magnetic	Electro-static	Electro-magnetic	E.M.U.
			Electro-static	E.S.U.
Quantity.......	$[M^{1/2}L^{1/2}\mu^{-1/2}]$	$[M^{1/2}L^{3/2}T^{-1}\kappa^{1/2}]$	$[L^{-1}T\kappa^{-1/2}\mu^{-1/2}]$	c
Current........	$[M^{1/2}L^{1/2}T^{-1}\mu^{-1/2}]$	$[M^{1/2}L^{3/2}T^{-2}\kappa^{1/2}]$	$[L^{-1}T\kappa^{-1/2}\mu^{-1/2}]$	c
Pot. diff........	$[M^{1/2}L^{3/2}T^{-2}\mu^{1/2}]$	$[M^{1/2}L^{1/2}T^{-1}\kappa^{-1/2}]$	$[LT^{-1}\kappa^{1/2}\mu^{1/2}]$	c^{-1}
Resistance.....	$[LT^{-1}\mu]$	$[L^{-1}T\kappa^{-1}]$	$[L^2T^{-2}\kappa\mu]$	c^{-2}
Capacity.......	$[L^{-1}T^2\mu^{-1}]$	$[L\kappa]$	$[L^{-2}T^2\kappa^{-1}\mu^{-1}]$	c^2
Inductance.....	$[L\mu]$	$[L^{-1}T^2\kappa^{-1}]$	$[L^2T^{-2}\mu\kappa]$	c^{-2}

The following table gives the practical units in terms of the corresponding units of both the electromagnetic and the electrostatic systems:

1 Ampere		10^{-1} E.M.U.'s	$= 3 \times 10^9$ E.S.U.'s
1 Coulomb		10^{-1} E.M.U.'s	$= 3 \times 10^9$ E.S.U.'s
1 Volt		10^8 E.M.U.'s	$\frac{1}{3 \times 10^2}$ E.S.U.'s
1 Ohm		10^9 E.M.U.'s	$= \frac{1}{9 \times 10^{11}}$ E.S.U.'s
1 Farad		10^{-9} E.M.U.'s	$= 9 \times 10^{11}$ E.S.U.'s
1 Microfarad	=	10^{-15} E.M.U.'s	$= 9 \times 10^5$ E.S.U.'s
1 Henry	=	10^9 E.M.U.'s	$\frac{1}{9 \times 10^{11}}$ E.S.U.'s

THE RATIONALIZED PRACTICAL SYSTEM OF UNITS

16. Advantages of the Rationalized System.—In the discussion of the practical system it was pointed out that our present working units are decimal multiples of the corresponding units of the electromagnetic system. In fixing these ratios the international conferences have selected values in such a way that the electrical quantities commonly measured are expressed by numbers of ordinary magnitude. This, in reality, constitutes a mixed system of units, and, as a result, many of the formulæ used in every day calculations contain factors such as 10^{-1}, 10^8, 10^9, etc. Again, a system based upon the unit magnetic pole and the unit electric charge as given in paragraphs 9 and 10, respectively, inevitably leads to many formulæ in which the factor 4π appears.

It has been pointed out by Perry[1] and by Fessenden[2] that by properly choosing new units for magnetomotive force and field strength, and by submerging the factor 4π in the arbitrary constants defining the dielectric and magnetic properties of materials, that these objectionable factors may be eliminated, and all that Heaviside sought to accomplish by his "Rationalized System of Units," realized. In an admirable paper entitled "A Digest of the Relations between the Electrical Units and the Laws underlying the Units," Bennett[3] has carried out the suggestions of Perry and Fessenden and has developed a consistent series of defining equations and working formulæ in which the objectionable factors are suppressed, and has clearly set forth the relations between the units of the different systems. In this treatment, a new unit of force, the "Dyne-seven" (equal to 10^7 dynes) has been introduced. The advantage of this unit is obvious, since it is the force that, when acting through one centimeter, performs one joule of work.

17. Definitions. 1. *Unit Quantity of Electricity.*—The method followed here is similar to that of the electrostatic system in that the unit of charge is taken as the fundamental unit, and its magnitude is arrived at by an application of Coulomb's law, namely,

$$F = \frac{Q_1 Q_2}{kd^2}$$

[1] PERRY, *Electrician,* vol. 27, 1891, p. 355.

[2] FESSENDEN, *Electrical World,* vol. 34, 1899, p. 901.

[3] Bull. 880, Univ. of Wisconsin.

Where Q_1 and Q_2 are two charges of electricity placed d cm. apart. The quantity k is a constant depending upon the medium in which the charges are placed, and for free space is arbitrarily put equal to $\frac{1}{9 \times 10^{11}}$. If $Q_1 = Q_2 = 1$ and $d = 1$, then $F = 9 \times 10^{11}$ dyne sevens. Accordingly the coulomb is that quantity of electricity which repels a similar quantity at a distance of one centimeter in a vacuum with a force of 9×10^{11} dyne sevens. The coulomb, as thus defined, is identical with that defined in the practical system of Art. 12.

2. *Permittivity of a Medium.*—The factor k in the expression for Coulomb's law is called the dielectric constant of the medium. Since in many formulæ the factor $\frac{k}{4\pi}$ appears, it is expedient to replace k by a new medium constant defined by the relation

$$p = \frac{k}{4\pi}$$

and Coulomb's law is then written

$$F = \frac{Q_1 Q_2}{4\pi p\, d^2}$$

The quantity p is called the permittivity of the medium, and for free space has the numerical value

$$p = \frac{k}{4\pi} = \frac{1}{4\pi \cdot 9 \times 10^{11}} = 8.84 \times 10^{-14}$$

The relative permittivity of a substance is the ratio of the permittivity of the substance to the permittivity of free space, and is thus numerically equal to the dielectric constant or specific inductance capacity as ordinarily defined.

3. *Unit of Current; the Ampere.*—A current of one ampere is flowing in a circuit if the quantity passing any plane in the circuit per second is one coulomb.

4. *Unit of Potential Difference; the Volt.*—A difference of potential of one volt exists between two points if the work required to carry one coulomb from one point to the other is one joule.

5. *Unit of Resistance; the Ohm.*—A conductor has a resistance of one ohm if a difference of potential between its terminals of one volt maintains a current of one ampere.

6. *Unit of Capacitance; the Farad.*—A condenser has a capacitance of one farad if a charge of one coulomb produces differences of potential between its plates of one volt.

7. *Unit of Inductance; the Henry.*—A coil has an inductance of one henry if a current through it, changing at the rate of one ampere per second, induces within it an E.M.F. of one volt.

8. *Line of Magnetic Intensity.*—By a line of magnetic intensity or a line of force in a magnetic field is meant any line which is traced out by the center point of a small plane direction-finding coil,[1] as the coil is moved in the direction pointed out by its normal axis. Such lines are always found to be closed loops, which either link with electric currents or pass through magnets.

9. *Magnetic Flux Density.*—The magnetic flux density, B, at a point in a magnetic field is defined as a vector quantity whose direction is the positive direction along the line of magnetic intensity passing through the point, and whose magnitude is equal to the force upon a straight wire one centimeter in length carrying a current of one ampere, the direction of the wire making a right angle with a line of magnetic intensity through the point.

Unit of Flux Density.—The Weber per square centimeter.—If a wire one centimeter in length carrying a current of one ampere, in a direction at right angles to the lines of magnetic intensity is acted upon by a force of one dyne seven, the flux density is one weber per square centimeter. One weber per square centimeter equals 10^8 gauss.

10. *Relation Between the Magnetic Flux Density and the Current Causing the Field.*—Experimental measurements show that at any point in a field, free from iron, the value of the magnetic flux density, B, is directly proportional to the value of the current producing the field. For the special case of an annular ring uniformly wound with a coil of N turns, carrying a current I, experimental measurements show that the lines of magnetic intensity are circles lying within the ring as illustrated in Fig. 57 and that the value of the flux density, B, is uniform along each circle and has the value

$$B = \mu \frac{NI}{L}$$

[1] A direction-finding coil is a small plane circular coil carrying a continuous current. The coil is so mounted on gimbals that its normal axis is free to take any direction. The normal axis is a line perpendicular to the plane of the coil at its center. The positive direction along the normal axis is defined to bear the same relation to the direction of the current around the coil that the direction of advance of a right-hand screw bears to its direction of rotation. This is called the right-hand screw convention.

in which L is the length of the circle. μ is a constant having the value 1.257×10^{-8} for all except ferromagnetic materials.

11. *Permeability.*—The constant μ, which appears in the equation expressing the relation between the flux density and the current, is called the permeability of the medium in which the magnetic field is set up. It is a constant analagous to conductivity in the conducting field and to permittivity in the electric field. This unit is called the weber per ampere turn per centimeter and is equal to $4\pi \times 10^{-9}$ units of permeability as defined in the unrationalized practical system.

12. *Magnetic Intensity.*—The defining equation of (10) may be written in the form

$$\frac{B}{\mu} = \frac{NI}{L}.$$

The expression $\frac{B}{\mu}$ appears in so many calculations dealing with magnetic fields that, for the sake of convenience, the name "magnetic intensity" or "strength of field" is given to it. It is seen to be equal to the number of ampere turns per centimeter. This unit of field strength is called the ampere turn per centimeter and is equal to $\frac{10}{4\pi}$ gilberts per centimeter.

13. *Magneto-motive Force.*—The line integral $\int Hdl$ for any closed magnetic circuit is called the magneto-motive force for that circuit. For the simple circuit of Fig. 57 we have $\int Hdl = HL = NI$. The unit of magneto-motive force is the "Ampere Turn" and is equal to $\frac{4\pi}{10}$ gilberts.

14. *Reluctance, Ampere Turn per Weber.*—A magnetic circuit possesses a reluctance of one ampere turn per weber if a magneto-motive force of one ampere turn produces a flux of one weber. One ampere turn per weber equals $\frac{10^9}{4\pi}$ oersteds.

CHAPTER II

GALVANOMETERS[1]

18. Description of a Galvanometer.—A galvanometer is an instrument for the detection and measurement of very small electric currents. Strictly speaking, when used merely for the detection of an electric current, as, for example, in determining the balance condition for a Wheatstone bridge or a potentiometer, it should be called a galvanoscope, and the term galvanometer restricted to the case in which it is standardized and used for the accurate measurement of currents. The fundamental principle upon which all galvanometers operate is the reaction between a current and a magnetic field, one of which is fixed and the other movable. There are two types of instruments, named after their originators, and known respectively as the Thomson and the D'Arsonval types.

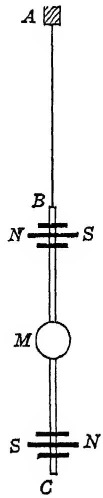

Fig. 5. Astatic needle system.

19. Thomson Galvanometer.—The Thomson galvanometer was invented by William Thomson (Lord Kelvin) and was first used as a detecting instrument in connection with the trans-Atlantic cable. It uses fixed coils and moving magnets, the axes of which are placed at right angles to the fields produced by the current in the coils. A high sensitivity requires, among other things, that the restoring torque on the moving system should be as small as possible. This is accomplished by use of the so-called astatic system which is illustrated in Fig. 5. A rigid rod, *BC*, usually a slender glass tube, is suspended by a very fine quartz fibre *AB*. This rod carries two systems of magnets *NS* placed with their planes accurately parallel, but with polarities reversed. If the magnetic moments of the two groups of magnets are equal, then when the system

[1] Laws, Electrical Measurements, chap. I.
Brooks and Poyser, Magnetism and Electricity, chap. XIX.
Hadley, Magnetism and Electricity, chap. XVI.

is placed in a uniform magnetic field, it will remain in any position in which it is placed, since the torque on one group of magnets is balanced by that on the other.

The fixed coils which carry the current to be measured are wound in opposite directions so that the reactions of their fields upon the magnets of the moving system give torques in the same direction. By making the system very light, e.g., a few milligrams, and by using a very fine quartz fibre for suspension, it is possible, with this type of instrument, to measure currents of the order of 10^{-12} amperes. Since the fields due to currents of such magnitudes are very weak, slight gradients in the external field produce relatively large differences in the torques upon the upper and lower magnet systems, and unsteadiness of the zero position results. Galvanometers of this type must, therefore, be carefully shielded magnetically.

Magnetic shields[1] may be either spherical or cylindrical in shape, but since no openings may be permitted without serious reduction in effectiveness, the latter form is usually employed. It has been found that if the iron is all concentrated in a single cylindrical shell having an outside diameter five times that of the inner, the effectiveness is 98 per cent of that of a shield having an infinite thickness. Furthermore, for a given amount of iron, the effectiveness is greatly increased by using several concentric cylinders. For extreme sensitivity, Thomson galvanometers are made very small, and the coils are often mounted in a solid iron container made by splitting a soft iron rod longitudinally and drilling small holes in each half to receive the coils.

20. D'Arsonval Galvanometer.—The D'Arsonval galvanometer consists of a fixed, permanent horse-shoe magnet and a light coil suspended between the pole pieces by a fine phosphor-bronze ribbon, the plane of the coil being parallel to the direction of the field. The current is led to the coil by the supporting ribbon and away by a helix of the same material attached at the bottom. While this type of instrument cannot be made as sensitive as the Thomson, it has the following special advantages: (*a*) The deflections are but little affected by variations in the external magnetic field; (*b*) the instrument may face in any direction; (*c*) the moving system may be made aperiodic, thus avoiding loss of time in waiting for it to come to rest. For these reasons, except where extreme sensitivity is required, the D'Arsonval

[1] Wills, *Physical Review*, vol. 24, 1907, p. 243.

galvanometer has practically replaced the Thomson for general laboratory work.

There are two distinct purposes for which galvanometers are used: (*a*) The measurement of small currents, and (*b*) the measurement of small quantities of electricity, such as are obtained by the discharge of condensers. When designed for the first purpose, they are called "current galvanometers" and for the second, "ballistic galvanometers."

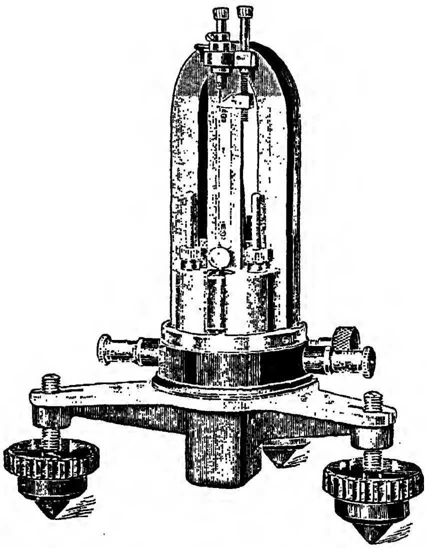

FIG. 6.—High sensitivity galvanometer with cover removed.

21. The Current Galvanometer.—Figure 6 shows a high sensitivity current galvanometer manufactured by the Leeds and Northrup Company. The permanent magnet is mounted in a vertical position and is provided with pole tips shaped so as to give a nearly cylindrical gap between them. Coaxial with this gap is placed a cylinder of soft iron and the coil rotates in the annular space thus formed. The suspension is carried on a rod supported by a bracket from the magnet. A set screw permits a vertical adjustment of the coil and the knurled head, which projects through the top of the case, gives a rough adjustment for zero position on the scale. A slow motion screw at the base of the instrument gives the final zero setting The axis of the coil

is made to coincide with that of the gap by means of three leveling screws which support the instrument. These screws are turned by heavy vulcanite nuts which give, at the same time, good insulation from ground. The right hand screw at the top operates an arresting device by means of which the weight of the coil may be taken off the suspension when the instrument is being moved. A cylindrical case, provided with a window to pass light to and from the mirror, protects the system against air currents.

22. Galvanometer Sensitivity.—If several galvanometers, selected at random, are connected in series and a definite current is sent through them, it will be found that there are marked differences in the response made by the individual instruments. Those showing greater responses are said to have higher sensitivities. The indication of a galvanometer is usually read by means of a beam of light reflected from a mirror, attached to the moving system, on a fixed scale. Obviously, for a given motion of the system, the indication will be proportional to the distance from mirror to scale, and so it is customary, when comparing galvanometers, to place the scale at a distance of one meter, and to read the deflection in millimeters. The sensitivity of galvanometers is defined in a number of ways among which the following are the most common:

(*a*) *Microampere Sensitivity.*—This is defined as the deflection in millimeters of a spot of light on a scale one meter from the mirror when the deflecting current is one microampere.

(*b*) *Microvolt Sensitivity.*—By this is meant the deflection in millimeters of a spot of light on a scale one meter from the mirror when an E.M.F. of one microvolt is impressed across the terminals of the galvanometer.

(*c*) *Megohm Sensitivity.*—By this is understood the number of megohms which must be placed in the galvanometer circuit in order that with an impressed E.M.F. of one volt there results a deflection of one millimeter on the scale whose distance is one meter.

The dependence of the sensitivities, as just defined, upon the constants of the instrument and the relations between them may be understood from the following considerations. It will be assumed that the coil is rectangular in shape and that it is so supported as to be capable of rotation about a vertical axis of symmetry. It will also be assumed that the field is radial,

uniform and horizontal, as shown in Fig. 7. A field of this sort is obtained by means of a cylindrical core between properly shaped pole pieces, and has the advantage that, for a constant current through the coil, the torque is independent of its angular position.

Let l be the length of the coil; b its width; n the number of turns; and H the strength of the field in which it is placed. Calling T the torque on the coil when the current flowing through it is i, we have, if *c,g,s* units are used,

$$T = Hinlb \tag{1}$$

The quantity nlb, that is, the product of the number of turns and the area of the coil, is frequently called the "equivalent winding surface." Designating this by E we have

$$T = HiE = Ci \tag{2}$$

where C, equal to HE, is the torque for unit current and is called the "Dynamic Constant" for the instrument.

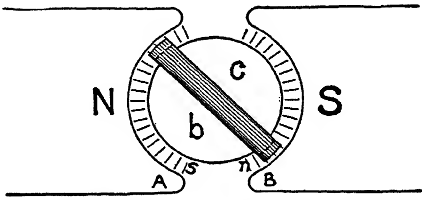

FIG. 7.—Diagram of moving coil galvanometer.

As the coil rotates, it twists the supporting metallic ribbon which exerts an elastic counter torque proportional to the angle of twist, and the coil takes an equilibrium position such that the two torques balance each other. Designating by τ the constant of the suspension, that is, the restoring torque when it is twisted through an angle of one radian, the angular deflection θ for a given current i satisfies the relation

$$\tau\theta = Ci \tag{3}$$

Letting A equal the angular displacement resulting from unit current we have

$$A = \frac{C}{\tau} = \frac{HE}{\tau} \tag{4}$$

A is the angular displacement in radians resulting from one C.G.S. unit of current and is, accordingly, the current sensitivity in C.G.S. units. The microampere sensitivity, as defined above, may be obtained from eq. (4) in the following manner: For

small deflections, the angular displacement of the system is proportional to the linear displacement of the spot of light along the scale. Moreover, the angular displacement of the reflected beam is twice that of the reflecting mirror. Accordingly, a deflection A in radians is equivalent to a deflection $2{,}000A$ when expressed as the deflection in millimeters of a spot of light along a scale at a distance of one meter from the mirror.

Again, if the current is measured in microamperes instead of C.G.S. units, it follows, since 1 microampere is 10^{-7} C.G.S. units, that the right-hand member of eq. (4) must be divided by 10^7 for this case. Therefore, replacing A by its value S divided by 2,000, where S is the deflection in millimeters due to one microampere, we have

$$S = \frac{C}{\tau} \times 2 \times 10^{-4} \tag{5}$$

This is the microampere sensitivity and is seen to be .0002 times the ratio of the dynamic constant to the suspension constant.

It is easily seen that the megohm sensitivity defined above is numerically equal to the microampere sensitivity just discussed. For, if S is the deflection in millimeters due to one microampere, then the current in amperes required for a deflection of one millimeter is $\frac{1}{S.10^6}$. Let M be the megohm sensitivity; that is, the number of megohms placed in series such that the deflection is 1 millimeter when the E.M.F. is 1 volt. By Ohms law,

$$i = \frac{1}{S.10^6} = \frac{1}{M.10^6} \text{ whence } M = S \tag{6}$$

To obtain the relation between the microampere and the microvolt sensitivity, let it be supposed that a difference of potential of 1 microvolt is impressed across the galvanometer. The current i in microamperes is given by

$$i = \frac{1}{R} \tag{7}$$

where R is the resistance of the galvanometer. The resulting deflection V, or the microvolt sensitivity is, accordingly,

$$V = Si = \frac{S}{R} \tag{10}$$

Thus the microvolt sensitivity is obtained by dividing the microampere sensitivity by the resistance of the coil.

23. Figure of Merit.—While the definitions of galvanometer sensitivity given above are convenient for distinguishing the

properties of one galvanometer from another, they are not well suited to the practical case in which the instrument is to be standardized and used for the measurement of currents. Here it is simpler to use the relation

$$i = \frac{\tau\theta}{C} = Fd \tag{11}$$

If d is the deflection in millimeters at a meter distance and i is in amperes, F is called the "figure of merit" or simply the "constant" of the galvanometer and is defined as the current in amperes required to produce a deflection of 1 millimeter at a distance of 1 meter. The smaller F, the greater is the sensitivity of the instrument.

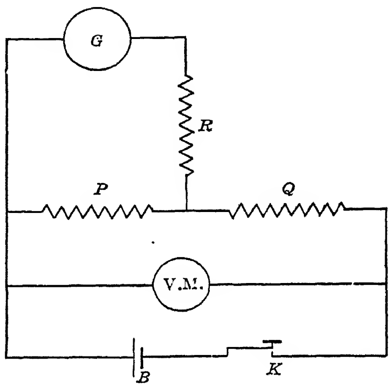

FIG. 8.—Connection for figure of merit.

To determine the figure of merit of a galvanometer it is merely necessary to pass known currents through the instrument and measure the deflections they produce. These currents may be supplied through a standardized variable resistance by a cell of known E.M.F., and computed by Ohm's law. Since, for most galvanometers, the required current is very small, the arrangement shown in Fig. 8 is generally employed. By making P small, usually 10 or 100 ohms, and Q large, 1,000 or 10,000 ohms, only a small fraction of the E.M.F. of the cell is effective in sending a current to the galvanometer G, and this current may be still further reduced by making R large. If $R + G$ is large in comparison to P, the fall of potential across P is

$$e = \frac{P}{P + Q}E \tag{12}$$

where E = E.M.F. of cell read by the voltmeter VM. The current i through the galvanometer is

$$i = \frac{e}{R + G} = \frac{P}{P + Q}\,\frac{E}{R + G} \tag{13}$$

and the constant F is given by

$$F = \frac{PE}{(P + Q)(R + G)}\,\frac{1}{d} \tag{14}$$

Since, in no galvanometer, is the deflection strictly proportional to current, it is necessary, in making a standardization, to use currents giving deflections over the entire range for which the instrument is to be used, determining from each a value of F which, when plotted as ordinates against d, as abscissas, gives a working curve showing F as a function of the deflections.

24. The Ballistic Galvanometer.—The ballistic galvanometer, which may be of either the moving coil or the moving magnet type, differs from the current galvanometer in that its moving system has a large moment of inertia, giving it a long period of vibration. If, while the system is at rest, a small quantity of electricity, such as a condenser charge, is suddenly passed through it, during the small interval of time that this electricity is flowing, there will be a torque acting on the system. This torque must be of very short duration as compared with the time required for the complete swing of the instrument, and is called an impulsive torque. The system is thus given an angular velocity, and an application of the laws of mechanics shows that the amplitude of the first ballistic throw is a measure of the impulsive torque applied, and hence of the quantity of electricity that has passed. The ballistic galvanometer is, then, an integrating rather than an indicating instrument. The rotational energy of the moving system is consumed in two ways: (*a*) The air surrounding the system is set in motion; (*b*) the relative motion of the coil and magnet induces a current in the coil, if the circuit is closed. Since the system is thus losing energy, each succeeding swing is less than the preceding one, the instrument comes gradually to rest, and the motion is said to be "damped." If the resistance across the galvanometer terminals is very large, the system will make several swings before coming to rest. If the resistance is small, the system will not vibrate at all, but will come to rest slowly. If, however, it is of the proper value, the motion may be just aperiodic; that is, it will not swing past zero, but will return to zero in the shortest time. The instrument is then said to be "critically" damped, and the resistance required is called the "critical resistance." In many instruments, the moving coil is wound on a closed copper form in which currents are induced as it swings, thus making it nearly aperiodic on open circuit.

25. Constant of a Ballistic Galvanometer.—A study of the equation of motion of the ballistic galvanometer shows that, no matter what the damping may be, whether zero or so great that

the motion is aperiodic, the first throw is proportional to the quantity of electricity discharged through it, the only limitation being that this discharge must take place before the system moves appreciably from its zero position. If the throw is small, so that the tangent is proportional to the angle, this fact may be expressed thus

$$Q = Kd \tag{15}$$

where Q is the quantity of electricity, d the deflection as read by a mirror and scale, and K a constant depending upon the sensitiveness of the instrument, numerically equal to the quantity necessary to give unit deflection. The smaller K, the greater is the sensitiveness of the instrument. If, then, K is known, we have a means of measuring small quantities of electricity

26. Theory of the Undamped Ballistic Galvanometer.—It will be assumed that the galvanometer is of the D'Arsonval type and that the field in which the coil moves is radial and uniform. It will also be assumed that the duration of the discharge is short compared to the time required for the first ballistic throw to take place. The conditions under consideration, then, are these: A small quantity of electricity, such as the charge of a condenser, is passed through the coil. While the current is flowing, the reaction between the current and the field produces a torque on the coil which starts it rotating. Although the duration of this torque is very short, the coil has, nevertheless, acquired a certain kinetic energy, and its motion is opposed only by the counter torque of the suspension, since we are neglecting damping. It will continue to rotate until its energy has been transferred to the suspension where it is stored as potential energy of elastic deformation. The coil then starts swinging in the reverse direction and when it passes through its zero position, it again possesses the same kinetic energy that it had originally, and will continue to oscillate indefinitely.

Let I be the moment of inertia of the coil, ω its angular velocity, α its angular acceleration, and θ its angular deflection at any instant. As in the discussion of the current galvanometer, let C be the coil constant, that is, the torque produced by unit current, and let τ be the suspension constant, that is, the counter torque for a twist of one radian. At any instant during the discharge, the equation of motion for the system is

$$Ci - \tau\theta = I\alpha \tag{16}$$

Since we are assuming that the discharge takes place before the coil swings appreciably from its zero position, the second term on the left hand side may be neglected; and, writing for α its value, $\frac{d^2\theta}{dt^2}$, we have

$$Ci = I\frac{d^2\theta}{dt^2} \tag{17}$$

Let t' be the time required for the discharge to take place. Then

$$C\int_0^{t'} idt = I\int_0^{t'} \frac{d^2\theta}{dt^2}dt \tag{18}$$

Carrying out this integration and letting ω' be the angular velocity at the time t', we have

$$CQ = I\omega' \tag{19}$$

Where Q is the quantity of electricity which passed through the coil. The kinetic energy thus acquired by the coil is

$$\text{Energy} = \tfrac{1}{2}I\omega'^2 = \tfrac{1}{2}\frac{C^2Q^2}{I} \tag{20}$$

If the coil swings through an angle θ_1, the potential energy of elastic deformation is

$$\overline{W} = \tau\int_0^{\theta_1}\theta d\theta = \tfrac{1}{2}\tau\theta_1{}^2 \tag{21}$$

Since this is equal to the initial kinetic energy of rotation, there results

$$C^2Q^2 \qquad \tau\theta_1{}^2$$

whence

$$Q = \frac{\sqrt{\tau I}}{C}\theta_1 \tag{22}$$

Inasmuch as the quantities in the coefficient of θ_1 are not readily determined, it is simpler to express this quantity in terms of the figure of merit of the galvanometer and its period of oscillation T. Since the coil executes an angular harmonic motion, its period is given by

$$= 2\pi\sqrt{\frac{I}{\tau}} \tag{23}$$

Substituting from (23) and (11) in (22) there results

$$Q = \frac{T}{2\pi}Fd \tag{24}$$

The constant K of eq. (15) is thus seen, for the undamped

ballistic galvanometer, to be $\frac{T}{2\pi}$ times its figure of merit when used as a current measuring instrument.

The ideal condition, i.e., zero damping, cannot be realized in practice. Moreover, it would be exceedingly cumbersome to use, because of the difficulty in bringing the coil to zero and maintaining it in this position while adjusting other parts of the apparatus in preparation for an observation. Since a certain amount of damping must necessarily be present, it is usually most convenient to increase the damping until the motion is just aperiodic. In this case, the galvanometer deflects to a certain point and then returns to zero in the quickest time; and, barring external disturbances, remains in this position indefinitely.

The theory of the damped ballistic[1] galvanometer is somewhat involved and is beyond the scope of this book. It may be shown, however, that when damping exists, the quantity of electricity passed through is given by

$$Q = \frac{T}{2\pi} F \left(1 + \frac{\lambda}{2}\right) d \qquad (25)$$

where λ is called the "logarithmic decrement" and is defined as the Naperian logarithm of the ratio of any deflection to the next one succeeding it in the opposite direction. It is thus seen that damping reduces the ballistic sensitivity of a galvanometer. Further, if the galvanometer is standardized under conditions such that the damping is different from what it is in use, the decrement must be determined in both cases and the difference allowed for by eq. (25).

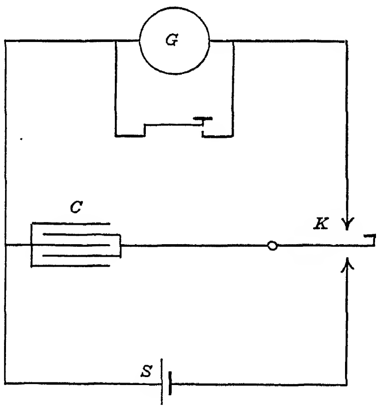

FIG. 9.—Condenser and standard cell method for obtaining constant of ballistic galvanometer.

27. Determination of the Constant of a Ballistic Galvanometer.—The standardization of a ballistic galvanometer consists in passing known quantities of electricity through it and measuring the deflec-

[1] O. M. STEWART, *Phys. Rev.*, vol. XVI, 1903, p. 158.
LAWS, Electrical Measurements, chap. II.

tions they produce. Two methods are in common use, known respectively as the "condenser and standard cell method" and the mutual inductance or "standard solenoid method."

1. *The Condenser and Standard Cell Method.*—This method consists in charging a condenser of known capacity by means of a standard cell, and then discharging this quantity through the galvanometer. The apparatus is arranged as shown in Fig. 9, where G is the galvanometer to be standardized, C a standard condenser, K a charge and discharge key, and S a standard cell. If V is the E. M. F. of the cell, the quantity stored in the condenser when the key is pressed down is

$$Q = CV \tag{26}$$

and since

$$Q = Kd \tag{27}$$

we have

$$K = \frac{CV}{d} \tag{28}$$

If C is a subdivided condenser, several different values should be used, a curve plotted using Q as abscissas and d as ordinates, and the constant computed from the slope of the straight line. If C is expressed in farads, V in volts, and d in centimeters, K will be given in coulombs per centimeter; but if C is in micro-farads, K will be given in micro-coulombs per centimeter.

2. *The Standard Solenoid Method.*—This method is especially applicable to cases in which the galvanometer is used on low resistance circuits where the damping is large. The known quantity of electricity discharged through the galvanometer is obtained from the secondary of a standard mutual inductance when a measured change in the primary current is produced. The connections are shown in Fig. 10 where AD is the primary of the mutual inductance, SS' the secondary coil, and G the galvanometer to be calibrated.

Let Q = quantity of electricity discharged through the galvanometer.
i = instantaneous current in galvanometer.
e = instantaneous E.M.F. in secondary coil.
I = value of primary current.
R = total resistance of secondary circuit.
M = mutual inductance between AD and SS.
T = time required for discharge to take place.

Then, from the above,

$$Q = Kd = \int_o^T idt \tag{29}$$

But

$$i = \frac{e}{R} \text{ and } e = M\frac{dI}{dt} \text{ from definition} \tag{30}$$

Hence,

$$Kd = \int_o^T \frac{M}{R}\frac{dI}{dt}dt = \frac{M}{R}\int_o^I dI = \frac{MI}{R} \tag{31}$$

or

$$K = \frac{MI}{R}\frac{1}{d} \tag{32}$$

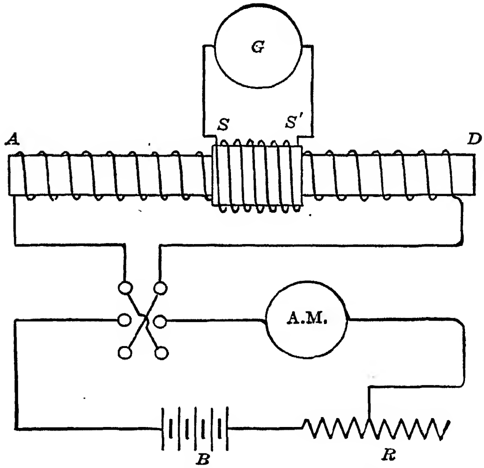

Fig. 10.—Standard solenoid method for ballistic galvanometer constant.

If one of the coils is uniformly wound and has a length great in comparison to its diameter, as the primary AD of Fig. 10, it is called a standard solenoid. The mutual inductance may then be calculated from the dimensions of the solenoid, and the number of turns on the coils, as follows:

Let A = area of standard solenoid
L = length of standard solenoid
N = number of turns on standard solenoid
H = field strength in standard solenoid
ϕ = total flux in standard solenoid
n = turns on secondary of standard solenoid.

The coefficient of mutual inductance may be defined, in electromagnetic units, as the number of magnetic linkages through the secondary when unit current is flowing in the primary, where, by linkages is understood the product of the number of turns and the total flux. As the secondary coil surrounds the standard solenoid, we have

$$M = n\phi = nHA \tag{33}$$

$$= \frac{4\pi NnA}{L} \text{ electromagnetic units} \tag{34}$$

Since, however, we wish M expressed in henries, we must divide by 10^9, the number of E.M.U's. required for one henry. Accordingly, our equation becomes

$$K = \frac{4\pi NnA}{RL10^9}\frac{I}{d} \tag{35}$$

It is customary to reverse the current through the primary of the standard solenoid instead of merely "making" it as implied in the above derivation. The limits of integration in equation (31) should then be $-I$ and $+I$ instead of O and I, in which case our formula becomes

$$K = \frac{8\pi NnA}{RL10^9}\frac{I}{d} \tag{36}$$

In the above derivation, we have assumed that the field strength at the center of the standard solenoid is given by the formula

$$H = \frac{4\pi NI}{10L} \tag{37}$$

which is true only for an infinitely long solenoid. If the length of the standard solenoid is fifty times the diameter, the error, which is due to the demagnetizing effects of the ends, is less than one-half of one per cent. We have further assumed that there is no magnetic leakage between primary and secondary coils, a condition which is never realized. Our value for M, computed above, is, therefore, too large; and for very accurate work, a correction should be made. If we call f the demagnetization and leakage factor, our corrected formula for K becomes

$$K = \frac{8\pi NnA}{RL10^9}\frac{I}{d}(1 - f) \tag{38}$$

In practice, it is customary to obtain a series of deflections using different values of I, then plot I as abscissas and d as ordinates, and obtain the ratio $\frac{I}{d}$ from the slope of the line. If

practical units of electrial quantities are used throughout, K will be expressed in coulombs per centimeter.

28. The Fluxmeter.—It was pointed out above, as a necessary condition that the ballistic galvanometer should give indications proportional to the quantity of electricity passed through it, that this passage must be completed before the moving system swings

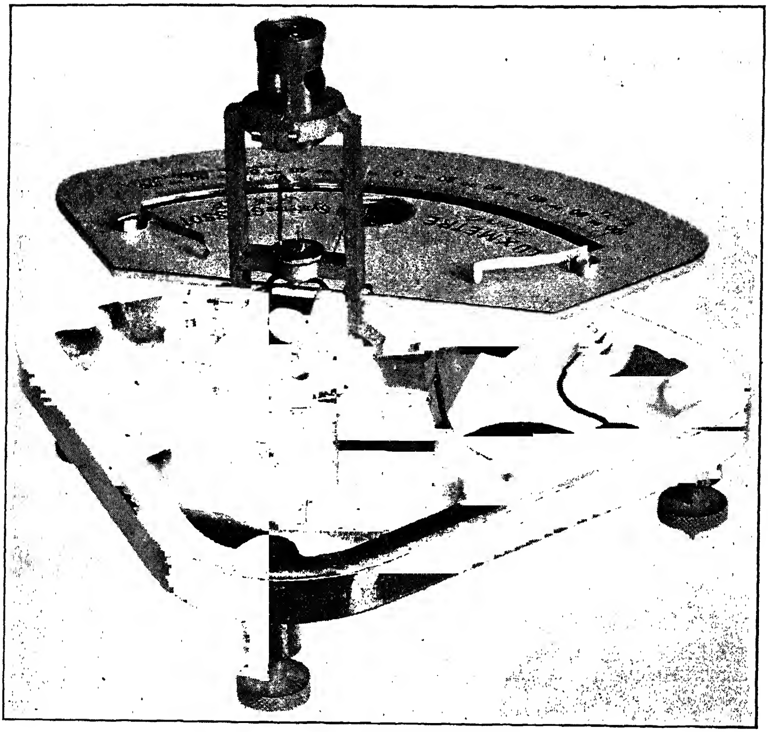

FIG. 11.—Grassot Flux Meter.

appreciably from its zero position. In certain instances, as, for example, the testing of iron possessing magnetic viscosity, the induced current which is passed through the galvanometer persists too long, and hence the ordinary instrument cannot be used. The Grassot fluxmeter is a modified ballistic galvanometer of the moving coil type, in which this difficulty is overcome. The coil is suspended by a fine silk fibre and is practically free from restoring forces, the current being led in and out by

means of fine helical springs. It is rectangular in shape and is placed in a field as nearly radial as possible, with respect to its axis of rotation, the parts involved being similar to those of the Weston ammeter. The torque, for a given current, is practically independent of the position of the coil. When connected to a resistance equal to or less than its critical resistance, the coil is stationary in any position. When a given quantity of electricity is discharged through it, it moves to a new position and the change in position is proportional to the quantity that passed, no matter how long a time was required. It is standardized and used as an ordinary ballistic galvanometer, except that some means must be provided for bringing it back to its zero position. Figure 11 shows the construction of an instrument of this type.

29. Theory of the Fluxmeter.[1]—As originally designed, the fluxmeter was intended as an instrument for the direct measurement of magnetic flux density. For this purpose, coils are constructed which consist of a definite number of turns wound on a plate of nonmagnetic material, the area of which must be carefully measured. These coils are made very thin so that they may be inserted in a narrow air gap such as exists between the armature and pole pieces of a dynamo. The measurement of an unknown flux density consists then in connecting the test coil by flexible leads directly to the fluxmeter and placing it at right angles to the flux to be measured. The instrument is brought to zero by some suitable device. The test coil is then withdrawn from the flux and the accompanying deflection of the instrument, multiplied by its constant, measures directly the change in flux through the test coil.

The direct proportionality between change of flux through the coil and deflection of the instrument may be shown as follows:

Let ϕ = flux through the exploring coil
N = number of turns in exploring coil
L = inductance of exploring and galvanometer coils
R = resistance of exploring and galvanometer coils
C = constant of galvanometer coil = $Hnlb$
I = moment of inertia of galvanometer coil

[1] Laws, Electrical Measurements, p. 124.
M. E. Grassot, Fluxmètre, *Journal de Physique*, 4th series, vol. 3, 1904, p. 696.

ω = angular velocity of galvanometer coil
i = instantaneous current in galvanometer coil
θ = angular deflection due to change of flux

As the test coil is withdrawn from the flux, there is induced in it an E.M.F. given by $\frac{d\phi}{dt}$. This is opposed by the counter E.M.F., $L\frac{di}{dt}$, due to the inductance of the galvanometer coil, and also by an E.M.F. $C\omega$ due to the motion of this coil through the field of the instrument. Accordingly, the current at any instant is

$$i = \frac{N\frac{d\phi}{dt} - L\frac{di}{dt} - C\omega}{R} \tag{39}$$

The motion of the coil is given by

$$Ci = I\frac{d\omega}{dt} = \frac{CN}{R}\frac{d\phi}{dt} \quad \frac{CL}{R}\frac{di}{dt} \quad \frac{C^2\omega}{R} \tag{40}$$

Integrating between the limits o and t, where t is the duration of the change of flux and consequent motion of the galvanometer coil, we have

$$\frac{CN}{R}\int_0^t\frac{d\phi}{dt}dt = I\int_0^t\frac{d\omega}{dt}dt + \frac{CL}{R}\int_0^t\frac{di}{dt}dt + \frac{C^2}{R}\int_0^t\omega dt \tag{41}$$

Remembering that at both limits the current and angular velocity are each zero and that $\int\omega\,dt = \theta$, we have

$$-\phi_1 = \frac{C}{N}\theta \tag{42}$$

The change of flux through the test coil is thus seen to be directly proportional to the angle θ through which the coil rotates. This deflection may be read either by a pointer or a mirror and scale. The fluxmeter may be used for almost any purpose for which the ballistic galvanometer is suited, but has, in general, a somewhat lower sensitivity.

30. Checking Devices.—If a ballistic galvanometer is not critically damped, it is convenient to have some device to check its motion and to set it accurately at its zero position. If the instrument is of the D'Arsonval type, this may usually be accomplished simply by a short circuiting key. However, since most keys possess slight thermal E.M.F.'s, the zero with the key closed will usually be different from the normal zero with the key open. When the galvanometer is used on a closed circuit, the

device shown in Fig. 12 is much more satisfactory. It consists of a coil of wire through which a bar magnet may be moved. The coil is connected in series with the galvanometer and the motion of the magnet induces in it a small E.M.F., positive or negative,

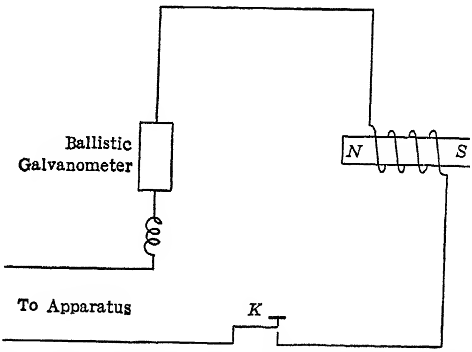

FIG. 12.—Checking device for ballistic galvanometer.

depending upon the direction of motion. The key must remain closed except when it is necessary to "get a new hold" on the galvanometer. With a little experience the instrument may, with this device, be set on zero very quickly and accurately.

CHAPTER III

MEASUREMENT OF RESISTANCE

31. Ohm's Law.—When a current of electricity is flowing from one point to another along a conductor, a difference of potential is found to exist between these points. The magnitude of the difference of potential depends upon the current and upon a property of the material in virtue of which it offers opposition to the passage of current. The relation between potential difference and current was first given by Ohm, and is known as Ohm's law. It states that, as long as the physical condition of a conductor remains unchanged, there is a constant ratio between the current and potential difference; or, in symbols,

$$I = \frac{E}{R} \tag{1}$$

where the proportionality factor R is called the resistance of the conductor. This law is a result of experiment and has been found to be true within the limits of the most refined measurements.

32. Specific Resistance.—For a uniform conductor, other conditions remaining the same, the resistance is proportional to the length and inversely proportional to the area of cross section. Hence, if l represents the length and a the cross section, we have

$$R = \rho \frac{l}{a} \tag{2}$$

where ρ is a constant depending upon the material of the conductor. Considering this as a defining equation for ρ, we see that, when l and a are unity, ρ equals R. The constant ρ is thus the resistance of a unit cube of the material, and is known as the Specific Resistance. In tabulating the resistivities of substances, the specific resistance is a convenient quantity to use, since knowing it, one can readily compute the resistance of a conductor of any length and cross section by means of eq. (2). The value of ρ depends upon the units employed for the measurement of length and resistance. Since the resistance of a unit cube of any metal is a very small quantity, it is customary to express the specific resistance in microhms per centimeter cube where a

microhm is one millionth of an ohm. Alloys, in general, have a much higher specific resistance than pure metals, and the presence of even a trace of another metal which, of itself, may be a good conductor, has a considerable effect upon the resistance; and hence, copper, for electrical purposes, should be pure.

33. Temperature Coefficient of Resistance.—The resistance of all conductors is found to change with the temperature. In the case of the pure metals, the resistance increases with increasing temperature, while for carbon and electrolytes, the opposite is true. The former are said to have a positive, and the latter a negative, temperature coefficient. Experiment shows that, over relatively large intervals of temperature, the resistance of a given conductor, at any temperature t, may be expressed by the equation

$$R_t = R_o(1 + \alpha t + \beta t^2 + \ldots\ldots) \tag{3}$$

where R_0 is the resistance at zero degrees and α and β are constants depending upon the material and the temperature interval concerned. Over small ranges of temperature, the change in resistance is nearly proportional to the change in temperature, and may be represented by the linear relation

$$R_t = R_o(1 + \alpha t). \tag{4}$$

The coefficient α is called the "Temperature Coefficient," and is the change in resistance per ohm per degree change in temperature. Some alloys, such as german silver and manganin, have a very small temperature coefficient, that of the latter being zero at some temperatures. Manganin is well suited, for this reason, for the construction of standard resistances.

34. Measurement of Resistance.—The independent or "absolute" determination of resistance, that is, measurement in terms of the fundamental units of length, mass, and time, is a matter of considerable difficulty; and so the establishment of primary standards is, at the present time, left almost entirely to government Bureaus of Standards, which are especially equipped for work of this character. On the other hand, the comparison of resistances, even to a high degree of accuracy, is relatively simple, and it is with work of this character only that we are concerned here.

35. The Wheatstone Bridge.—This is the usual method employed for comparing resistances of ordinary magnitudes, and its principle may be readily understood from Fig. 13. Four resistances are connected in the form of a diamond, with current

from the battery entering at A, where it divides in two parts which unite again at B. The galvanometer G is connected across the other corners of the diamond.

Since the points P and Q possess potentials intermediate between those of A and B, it must be possible to make Q have the same potential as P by suitably choosing R_3 and R_4. When this condition has been established, no current flows through the galvanometer, as indicated by zero deflection, and the bridge is

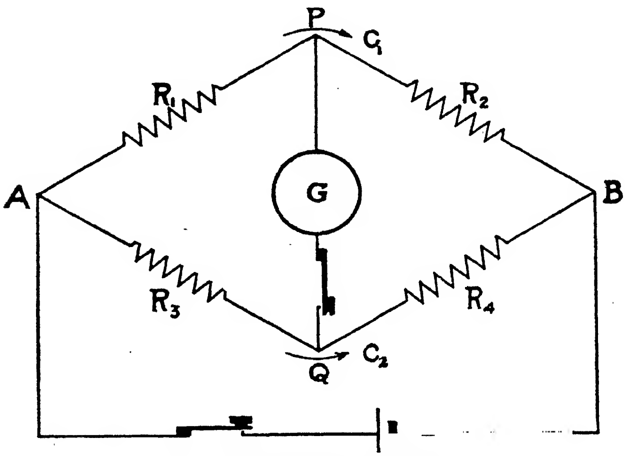

FIG. 13.—Wheatstone Bridge.

said to be balanced. Calling the current through R_1 and R_2, C_1, and that through R_3 and R_4, C_2, we have the

P.D. between A and P = P.D. between A and Q and
P.D. between P and B = P.D. between Q and B.

By Ohm's law

$$R_1C_1 = R_3C_2 \tag{5}$$

and

$$R_2C_1 = R_4C_2 \tag{6}$$

Whence

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \tag{7}$$

This is the law of the Wheatstone bridge, and it is clear that, if three of these resistances are known the fourth may be computed.

36. The Slide Wire Bridge.—If, in the above equation, R_1 is an unknown and R_2 a standard resistance, the former may be

expressed in terms of the latter by means of the ratio of R_3 to R_4. It is obvious then that the actual values of R_3 and R_4 need not be known, their ratio being sufficient. Advantage is taken of this fact in the construction of the slide wire bridge, which is shown diagrammatically in Fig. 14 where the corresponding points of Fig. 13 are indicated by the same letters. R_3 and R_4 are replaced

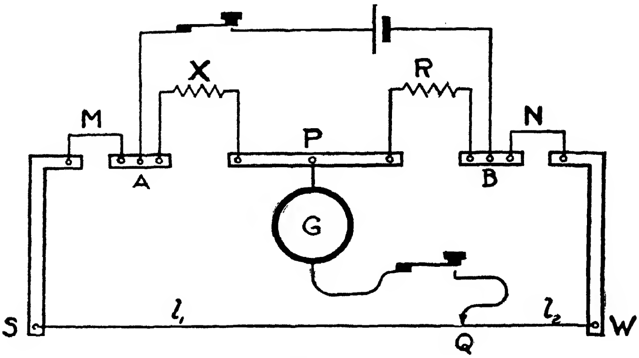

FIG. 14.—Slide wire bridge.

by portions of the slide wire SW and their magnitudes varied by moving the slider Q. Calling ρ the resistance of 1 cm. of the wire, we have

$$\frac{X}{R} = \frac{\rho l_1}{\rho l_2} \tag{8}$$

whence

$$\frac{l_1}{l_2}R. \tag{9}$$

In order to increase the accuracy of setting, and to reduce the relative errors in measuring l_1 and l_2, especially where X and R have quite different values, resistances are introduced in place of the links M and N, which may be measured in terms of ρ and expressed, therefore, as a certain number of slide wire units to be added to l_1 and l_2.

37. The Post-office Box.—A more compact form of Wheatstone bridge is shown in Fig. 15, which is known as the post-office box, from the fact that it was adopted at an early date by the British Post and Telegraph Office. The slide wire is replaced by two series of ratio coils, AQ and BQ, having resistances of 10, 100, 1,000 and 10,000 ohms each, while the third arm is a series

of coils, arranged as in the ordinary resistance box, frequently having a total of 100,000 ohms. The unknown X is connected between B and P. Since the ratio of X to R is thus a decimal number, no calculation is required. With the ratio coils set at

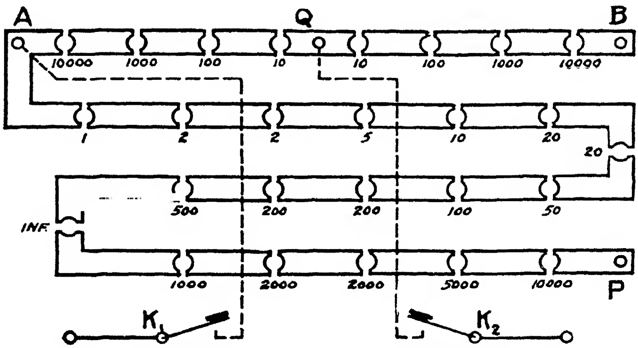

FIG. 15.—Post-Office box diagram.

1,000:1, resistances up to 100 megohms may be measured; while with the ratio reversed, resistances of the order of .001 may be detected. The range is thus great and its advantages are obvious. In using the box bridge, one should first use a 1:1 ratio, setting the coils at 100 ohms each, and obtain a rough balance, thus finding

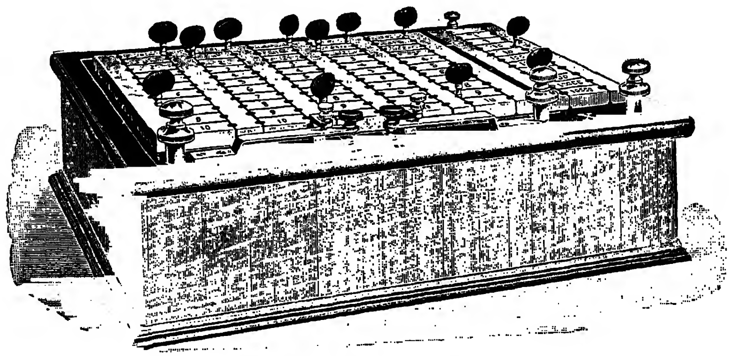

FIG. 16.—Post-office box.

the order of magnitude of the unknown. He should then choose such a ratio as will cause the balance setting of R to be as large as possible. For example, suppose C is found to be of the order of 45 ohms. By using a ratio of 1:1,000 a balance may be obtained

at 45,638, let us say, giving, as the value of the unknown, 45.638; while if a ratio of 1:100 had been used, the result would have been 45.64. The higher ratio thus increases the accuracy. Box bridges of the better class are provided with plugs for interchanging the ratio arms, by means of which inequalities in the internal connections of the bridge may be eliminated, and a check obtained upon the accuracy of the ratio coils. For accurate work, one should reverse the battery terminals in each case and re-balance, thus eliminating errors due to thermal and contact differences of potential. A convenient form of post-office box is shown in Fig. 16.

38. Measurement of Low Resistance.[1] *Kelvin's Double Bridge.*—For the measurement of extremely low resistances such as that of a few feet of trolley wire, cable, bus-bars, etc., the Wheatstone bridge is unsuited for two reasons: First, when the resistances to be compared are very low, the bridge becomes insensitive; and second, some sort of connectors must be used for joining the unknown to the bridge, and these may have a resistance comparable to that to be measured. The Kelvin double bridge avoids both of these difficulties. The general scheme of this circuit is shown in Fig. 17, where X and S are the unknown and standard resistances, respectively, through which a large current flows which need not necessarily be constant. There are four ratio coils, a, b, c, and d, arranged in pairs, while the galvanometer is connected at the points C and D, between each pair. By properly adjusting the ratio coils, C and D may be brought to the same potential, when no current flows through the galvanometer and the currents in X and S are equal. When the balance has thus been obtained, let us call I the current through X and S, I_1 that through a and c, and I_2 that through b and d. Then, by Ohm's law,

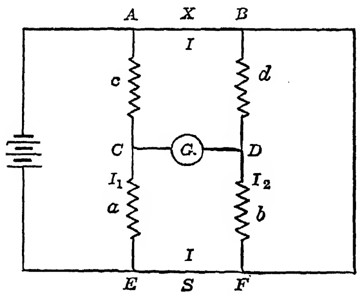

FIG. 17.—Diagram for Kelvin's double bridge.

$$cI_1 = XI + dI_2 \text{ and } aI_1 = SI + bI_2 \tag{10}$$

[1] NORTHRUP, Measurement of Resistance, chap. VI.
LAWS, Electrical Measurements, chap. IV.

Whence

$$XI = cI_1 - dI_2 \text{ and } SI = aI_1 - bI_2 \tag{11}$$

$$XI = c\left(I_1 - \frac{d}{c}I_2\right) \quad SI = a\left(I_1 - \frac{b}{a}I_2\right) \tag{12}$$

By the construction of the instrument,

$$\frac{d}{c} = \frac{b}{a}$$

which gives, on dividing equations (12),

$$\frac{X}{S} = \frac{c}{a} \tag{13}$$

which is the working formula for the instrument. The wiring diagram for one form of this bridge is shown in Fig. 18 where the points corresponding to those of the schematic diagram

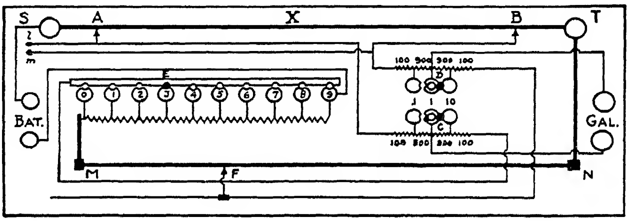

FIG. 18.—Laboratory form of Kelvin's double bridge

of Fig. 17, are lettered similarly. The unknown is represented as a heavy rod with potential taps at A and B, while the standard consists of the bar MN and the coils with posts numbered 0–9. Each of the coils, as well as the standard bar, has a resistance of .01 ohms, and the resistance being used as the standard S, is that between the slider F and the movable plug E. The standard thus has a range of 0 to .1 ohms by infinitesimal steps. The ratio coils are situated to the right of the standard coils and are connected to the galvanometer in different ways by means of the plugs C and D. A little study of the connections will show that three different ratios are possible; namely, 1:10, 1:1, and 10:1. The plugs C and D must be placed opposite one another, since a double ratio must be maintained as indicated by the equations; that is,

$$\frac{X}{S} = \frac{c}{a} = \frac{d}{b} \tag{14}$$

The resistance X is that portion of the rod between the points A and B only. When the resistance to be determined is of some other form than a rod, it must be provided with two sets of leads; a heavy pair for the current, which are joined to the bridge at S and T, and a light pair for the potential drop across it, joined at l and m. The bridge thus measures the resistance of the conductor between the points to which the potential leads are attached.

The Leeds and Northrup form of self-contained Kelvin Bridge is shown in Fig. 18*A*. The low-resistance standard has nine divisions of .01 ohm each and a graduated bar, also of .01 ohm resistance. There are seven pairs of ratio coils varying from 0.1 to 10, the position of the selecting switches for which are indicated by the two dials near the right-hand end of the slider. The instrument does not include mountings for the unknown resistance indicated by ST of Fig. 18. Binding posts with appropriate markings are provided by means of which the unknown may be joined in series with the bridge and the potential leads joined to the ratio coils *cd*.

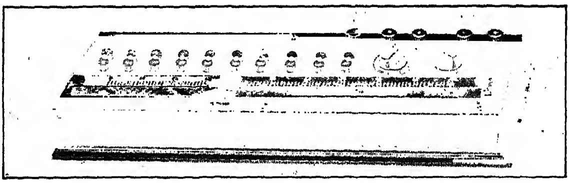

Fig. 18*A*.—Self-contained Kelvin bridge.

39. Experiment 1. *Specific Resistance of Materials.*—In this experiment, the specific resistance of three metals, copper, brass, and iron, is to be found. The metals are provided in the form of rods, which are to be clamped in the bridge at S and T. Make sure that good contact is obtained at A and B by polishing the bars at those points with emery cloth. Use, as a current supply, a ten-volt storage battery and include an ammeter and a reversing switch in this circuit, and a press key in the galvanometer circuit. Operate the bridge on 3 amperes. Measure the resistance of 50 cm. and 100 cm. lengths of each bar, reversing the current at each setting to eliminate errors due to thermal and contact potential differences within the instrument. Make at least four balances for each length approaching the balance point from both sides. Determine the diameter of the rod by means of

a micrometer gauge, taking the average of ten measurements uniformly distributed over its length.

Report.—1. Compute the specific resistance of each material in michroms per centimeter cube.

2. Compare your results with the data given in one or two of the standard tables of physical constants to be found on the reference shelves. How do you account for the discrepancies?

40. Carey-Foster Method for Comparing Two Nearly Equal Resistances.—A very accurate method for comparing two resistances which are nearly equal to one another has been devised by Carey Foster.[1] It possesses the advantage that

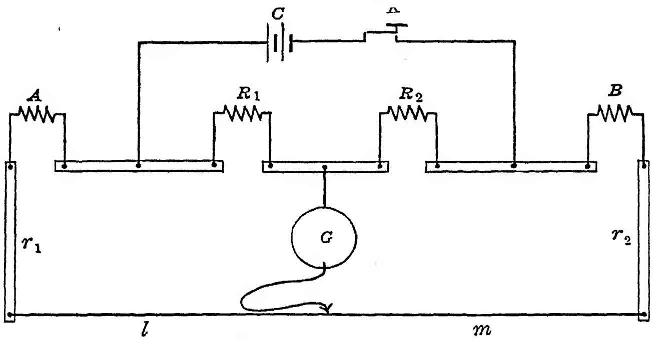

FIG. 19.—Wiring diagram for Carey-Foster bridge.

errors arising from the resistance of leads within the bridge, as well as those due to thermal and contact electromotive forces, providing they remain constant, are automatically eliminated. The wiring diagram is shown in Fig. 19. It consists of a slide wire bridge in which R_1 and R_2 are ratio coils and A and B are the resistances to be compared. Let r_1 and r_2 be the resistances of the internal bridge leads between the battery and slide wire connections on each side. Then if l_1 and m_1 are lengths of the bridge wire at balance, we have

$$\frac{R_1}{R_2} = \frac{A + r_1 + \rho l_1}{B + r_2 + \rho m_1} \tag{15}$$

where ρ is the reistance of the slide wire per unit length. Now let A and B exchange places, and let l_2 and m_2 be the corresponding lengths for a new balance. Then

$$\frac{R_1}{R_2} = \frac{B + r_1 + \rho l_2}{A + r_2 + \rho m_2} \tag{16}$$

[1] *Phil. Mag*, May, 1884.

Equating the right hand members of equations (15) and (16) and taking the resulting equation by addition, we have

$$\frac{A + r_1 + \rho l_1 + B + r_2 + \rho m_1}{B + r_2 + \rho m_1} = \frac{B + r_1 + \rho l_2 + A + r_2 + \rho m_2}{A + r_2 + \rho m_2} \quad (17)$$

Since $l_1 + m_1 = l_2 + m_2$, the numerators of these fractions are equal; the denominators are therefore also equal, whence

$$B + r_2 + \rho m_1 = A + r_2 + \rho m_2$$

$$A - B = \rho(m_1 - m_2) = \rho(l_2 - l_1) \quad (18)$$

The difference in the resistance of the two coils, A and B, is thus seen to be equal to the resistance of the slide wire between the

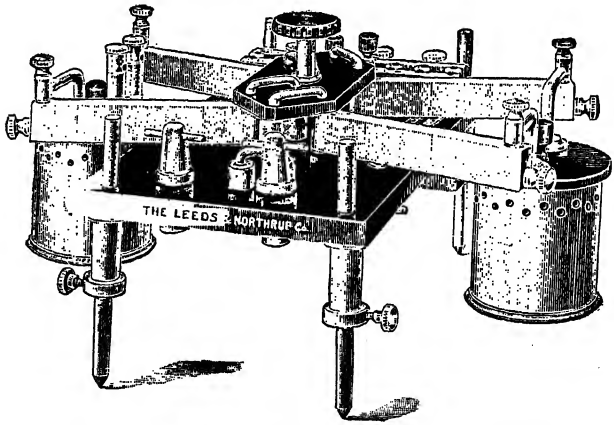

FIG. 20.—Coil interchanger for Carey-Foster bridge.

two points of balance, before and after the interchange of the coils. It is to be noted that this result is independent of the values of R_1 and R_2.

The Carey-Foster bridge is usually employed for the comparison of coils whose temperature must be maintained constant and they are usually immersed in some sort of oil bath for this purpose. A convenient device therefore must be provided for interchanging them without removing them from their baths or producing any changes in contact resistances by handling them. Figure 20 shows an arrangement for this purpose. The coils are supported at the ends of heavy copper bars which swing so as to receive units of different sizes. Contact between resistance terminals and bars as well as between bars and links of the com-

mutator are made by boring cups in the bars and partially filling them with mercury. The interchange of the coils is effected by a half turn of the commutator at the center. Adjustable legs enable the coils to be lowered in the baths to the proper depth.

To adapt the bridge to the comparison of coils of high as well as low resistances and to secure at the same time a satisfactory sensitivity, it is important to have several slide wires of different resistances per unit length. Figure 21 shows a complete bridge in which any one of three slide wires may be used at will. To obtain the effect of a very low resistance slide wire, one of ordinary magnitude may be shunted. A link, seen at the front of the switch board is provided for this purpose.

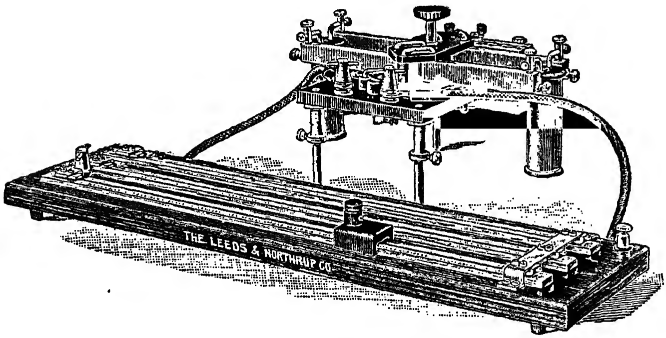

FIG. 21.—Complete Carey-Foster bridge.

41. Determination of ρ.—The Carey-Foster method requires that the slide wire be of uniform resistance, and that its resistance per unit length be accurately known. To measure ρ, the process of measurement above described may be inverted, using for A and B two equal coils of known resistance, one of which is shunted by a known variable resistance. By choosing appropriate values for the shunt, any desired difference between A and B may be obtained, and by changing R_1 and R_2 the balance points may be shifted to different positions along the slide wire. The constant ρ is obtained by substituting in eq. (18).

42. Experiment 2. *Measurement of Temperature Coefficient.*—The Carey-Foster method is particularly well adapted to the measurement of the variation of a resistance with temperature. The process consists in determining the difference between the resistances of two coils one of which is constant while the other is changed by holding it at different temperatures. The metal,

whose coefficient is to be measured, is in the form of a wire wound on a frame which may be placed in an oil bath to secure a uniform temperature throughout. Place the container in an ice pack and measure the resistance at a temperature as near as possible to 0°C. Next place the container in a water bath heated by an electric heater. Obtain the resistance at 10° intervals up to 80° C. While each measurement is in progress, remove the heater and place the bath upon a wooden stand. Stir the oil continuously and read the thermometer frequently. Settings should be made as rapidly as possible to avoid temperature changes. After the highest temperature has been reached, allow the bath to cool and check the readings at three points on the way down. The standard resistance should also be placed in an oil bath and its temperature maintained constant.

Report.—1. Plot a resistance temperature curve using resistance as ordinates and temperature as abscissas. Draw a straight line through these points to strike an average, and from it determine the values for R_0 and R_{85}. Compute the temperature coefficient α from the equation

$$R_t = R_0(1 + \alpha t)$$

2. Consult a table of physical constants and see if you can identify the wire tested from the value of the temperature coefficient obtained.

43. The Measurement of High Resistance.[1]—In previous sections, methods for measuring resistances of ordinary magnitude and for very small resistances have been considered. The measurement of very high resistances, such as the insulation between the bus-bars of a switch board and the ground, the armature bars and core of an electrical machine, insulation of cables, etc., requires special consideration. A ready method commonly employed by engineers, which gives reliable results for resistances up to several tenths of a megohm, and even higher, is that in which a voltmeter of known resistance is employed, the unknown high resistance taking the place of the multiplier in the ordinary use of the instrument. Suppose a voltmeter, of resistance r, is connected across a source of E.M.F., and the voltage, which we will call V, is measured. Then let an unknown resistance R be connected in series with the instrument across the same source. Since the voltmeter measures

[1] Northrup, Measurement of Resistance, chap. VIII.
Carhart and Patterson, Electrical Measurements, p. 92.
Gray's, Absolute Measurements in Electricity and Magnetism, p. 253.

the fall of potential across its own internal resistance, which we will call V_r, while the total voltage across R and r is that originally measured, *i.e.*, V, we may write, by Ohm's law,

$$\frac{V_r}{V} = \frac{r}{r + R} \tag{19}$$

Or,

$$V - V_r \qquad R \tag{20}$$

Whence

$$R = \frac{r(V - V_r)}{V_r}. \tag{21}$$

If the resistance R is too large, V_r will be insignificant compared with V, and the method evidently will not yield satisfactory results. For resistances of such magnitude, e.g., several megohms, recourse is generally taken to some leakage method in which the high resistance is used as an insulator, and its magni-

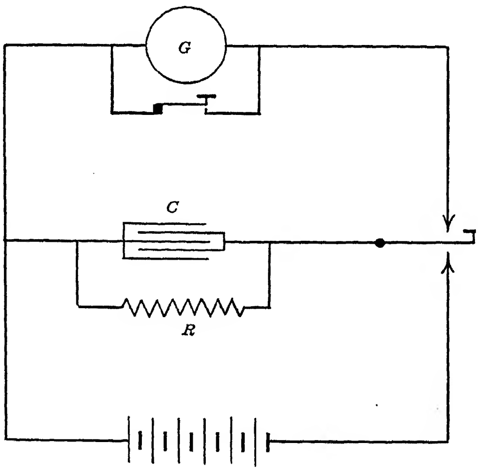

FIG. 22.—Insulation resistance by leakage.

tude estimated from the rate at which a known charge leaks through it. As an example, consider the case in which it is desired to measure the resistance of the insulation of a given length of cable. The cable should be coiled up and placed in a tank of water, both ends being left outside. This arrangement may be considered a condenser, one plate of which is the water, the other the wire, while the insulation is the dielectric. Its

electrical equivalent is shown in Fig. 22. If the wire and water are charged to a given potential difference and the insulation were perfect, the charge would remain constant; but, if the insulation possesses a slight conductivity, the charge will gradually leak through, reducing the potential difference of the condenser. The rate of leak may be estimated by measuring the residual charge after leakage has been going on for a definite time and comparing it with the original charge. The resistance is then calculated as follows:

Let C = capacity of the coil
V_0 = applied voltage
R = insulation resistance in ohms
V = instantaneous difference of potential
I = instantaneous leakage current
Q = instantaneous charge

The charge Q is given by

$$Q = CV \tag{22}$$

and

$$I = -\frac{dQ}{dt} = -C\frac{dV}{dt} = \frac{V}{R} \tag{23}$$

or

$$C\frac{dV}{dt} + \frac{V}{R} = 0 \tag{24}$$

Separating the variables

$$\frac{dV}{V} + \frac{dt}{CR} = 0 \tag{25}$$

Integrating

$$\log_e V + \frac{t}{CR} = K \tag{26}$$

where K is a constant of integration to be determined from the initial conditions. For this purpose, reckoning time from the instant when the leakage begins, the condition to be satisfied by the equation is, when $t = O$, $V = V_o$. Substituting these values in eq. (26), we have $\log V_o = K$. Replacing K by this value, we have

$$\log_e V + \frac{t}{CR} = \log_e V_o \tag{27}$$

or

$$\log_e V_o - \log_e V = \frac{t}{CR} \tag{28}$$

Thus,

$$\log_e \frac{V_o}{V} = \frac{t}{CR} \tag{29}$$

Solving,

$$R = \frac{t}{C \log_e \frac{V_o}{V}} \tag{30}$$

44. Experiment 3. *Insulation Resistance by Leakage.*—Connect the apparatus as shown in Fig. 22 where G is a ballistic galvanometer, C the coil under test, B a storage battery, and K a well insulated charge and discharge key. B should have such a voltage that the first throw of the galvanometer is about 15 cms. Since this is a leakage experiment, its success depends upon having all parts well insulated; the tank should be placed upon a glass plate or an insulated stand, and care be taken that no wires touch each other, the table, or other apparatus. Since the capacity of the coil does not change with time, the deflections of the galvanometer are proportional to the voltage across its terminals. Charge and discharge immediately thus obtaining a deflection proportional to V_o. Repeat this several times and take the average. Then charge for 10 seconds and place the key on the point marked insulate and, after allowing 15 seconds for leakage, again discharge and obtain a deflection. Repeat the operation for the following times of leak, 0.5, 1, 2, 5, 7, 10, 20 minutes. Now allow the condenser to charge for the above times, place the key on insulate for 10 seconds in each case and get a series of deflections on discharging. If common logarithms are used in computing the resistance, eq. (30) above becomes

$$R = \frac{t}{2.303\ C \log_{10} \frac{d_o}{d}} \tag{31}$$

If t is in seconds and C is in microfarads, R will be in megohms. The capacitance C of the cable may be obtained from the relation

$$Q_0 = CV_0 = kd_0 \tag{32}$$

where k, the constant of the galvanometer, is to be obtained by charging a standard condenser with a standard cell and discharging through the galvanometer, as explained in Art. **27.** Measure V_0 by an ordinary voltmeter. Measure the resistance of the wire of the cable by the voltmeter-ammeter method, using for this purpose about 20 amperes. Determine also the diameter of the wire by means of a micrometer gauge.

Report.—1. Compute the resistance of the cable for each time of leak from eq. 31, and plot the insulation resistance in megohms as ordinates, and time of leak as abscissas. It will be found that the resistance is not constant but increases with the time during which it was subjected to a voltage, approaching assymptotically to a limiting value. This is characteristic of all insulators of this class, and, in stating their resistances, the time for which it was determined must always be specified.

2. From the data on the resistance and diameter of the wire, find, by means of a wire table, the length of the cable, and compute the insulation resistance per mile for some selected time of leak. In making this calculation, remember that the insulation resistance is measured in the direction in which the leakage current flows, namely, radially from the wire to the outside, and that the longer the cable the less will be the total insulation resistance.

45. The Internal Resistance of Cells.[1]—It is a well-known experimental fact that when a cell is delivering current, the P.D. across its terminals is not the same as on open circuit but changes with the current, being less the larger the current. This is true not only for cells, but for all electrical generators containing internal resistance. Let a cell, having an E.M.F. of E volts and an internal resistance of r ohms, be connected to an external resistance of R ohms, and let I be the amperes flowing; then the rate at which energy is delivered by the cell is EI watts. Since the current must flow, not only through the external resistance, but also through the internal resistance of the cell, this energy will be consumed by both of these resistances; I^2R watts in the former, and I^2r watts in the latter. Accordingly we have

$$EI = (R + r)I^2 \tag{33}$$

or

$$E = RI + rI \tag{34}$$

This is an equation of E.M.F.'s which states that the total E.M.F. of the cell is equal to the external plus the internal potential drops. Putting the terminal P.D. equal to E', we have

$$E - E' = rI \tag{35}$$

from which it is seen that the internal resistance may be computed if E, E', and I are known. In fact, this is the method

[1] Northrup, Measurement of Resistance, chap. XI.
Carhart and Patterson, Electrical Measurements, pp. 96–105.

generally employed for cells which are able to furnish a considerable current without polarization; for example, storage cells. Suppose such a cell, whose internal resistance is to be measured is connected as shown in Fig. 23, where *AM*, *R*, and *K* are an ammeter, rheostat, and key, respectively. Let the voltmeter (V.M.) be an instrument taking no current, e.g., an electrostatic voltmeter, having an infinite resistance. When *K* is open, the voltmeter registers the total E.M.F. of the cell because there is no fall of potential across *r*, as no current is flowing. When *K* is closed, however, the voltmeter registers, not the total E.M.F. as before, but the terminal P.D. $E' = E - rI$, the portion rI being consumed in sending the current through *r*. Reading now the current, the internal resistance may be computed from Eq.

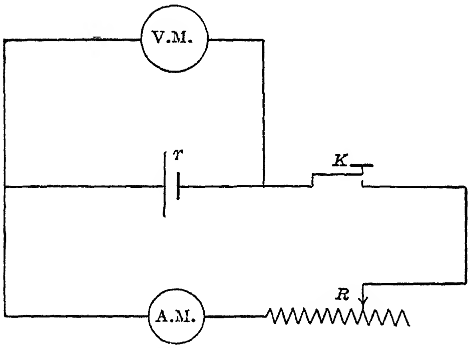

FIG. 23.—Voltmeter-ammeter method for internal resistance of cells.

35. In practice, an ordinary Weston voltmeter may be used without appreciable error since the resistance of the voltmeter is very large compared with that of the cell, and the *ir* drop within the cell, which is the quantity by which the indications of the instrument differ from the total E.M.F., is so small that it may be neglected, *i* being the current taken by the voltmeter. If, however, the cell is one that polarizes rapidly, this method cannot be used, since *E* and *E'* will depend upon how long the current has been flowing. This difficulty may be overcome by using a known resistance *R* and taking the voltages so quickly that little or no polarization sets in. The current *I* is given by

$$I = \frac{E}{R + r} = \frac{E'}{R}. \qquad (36)$$

Substituting either of these values for I, preferably the latter, in eq. (35), we have

$$E - E' = E'\frac{r}{R} \tag{37}$$

and solving for r, we have

$$r = R\frac{E - E'}{E'} \tag{38}$$

Since the right-hand member of this equation contains a ratio of voltages, it is not necessary to know actual values, relative values being sufficient; hence, any device giving indications proportional to the voltage may be used in place of the voltmeter; for example, a condenser and ballistic galvanometer.

46. Condenser and Ballistic Galvanometer Method.—The basis for this method is that the first throw of the galvanometer is proportional to the quantity of electricity discharged through it, and that the charge of a condenser is proportional to the potential difference across its terminals; that is

$$Q = CE$$

where C is the capacity of the condenser. Accordingly, if the voltages E and E' are used to charge the condenser, and these charges are then passed through the ballistic galvanometer, the deflections are proportional to the voltages; that is

$$Q = K_1d = CE$$

or

$$E = k_2d \tag{39}$$

Substituting in (38), we have

$$r = \frac{R(E - E')}{E'} = R\frac{(d - d')}{d'} \tag{40}$$

47. Experiment 4. *Internal Resistance of a Cell by Condenser Method.*—Connect the apparatus as shown in Fig. 24, where B is the cell to be tested, C a condenser, G a ballistic galvanometer, K_2 a charge and discharge key, and R a known variable resistance. First, with K_1 open, press down K_2 thus charging the condenser to the total E.M.F. of the cell, and discharge by allowing K_2 to rise, obtaining a deflection proportional to E. Take several readings in this manner and average. Then, having set R at a suitable value, close K_1, charge and discharge as above, opening K_1 as quickly as possible to avoid polarizing the cell. The average of several readings taken in this manner measures E', whence r may be computed. It is well first to practice operations upon a cell other than the one to be tested, in order to become expert in

manipulating the keys quickly and in their proper order. In carrying out this experiment, use the following values for R: 10, 7, 5, 4, 3, 2, 1, .5 and .2 ohms.

The current from the cell is given by—

$$I = \frac{E}{R + r} \tag{41}$$

where E is the total E.M.F. To obtain E, it is necessary to determine the voltage constant of the condenser and galvanometer system, which is, in reality, the constant of eq. 39. In other words, we must measure the voltage required to give unit deflection. For this purpose replace B by a standard cell and, with K_1 open, charge and discharge several times. Substitute in eq. (39) and solve for k_2.

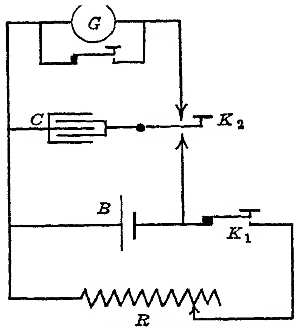

FIG. 24.—Condenser method for internal resistance of a cell.

Report.—1. Compute the internal resistance for each different current drawn from the cell and plot the former as ordinates against the latter as abscissas.

2. How do you account for the fact that the internal resistance is not constant?

48. Battery Test.—When a primary battery is furnishing current, it polarizes; that is, hydrogen, which is one of the products of the reactions going on within, collects on the positive plate. This, together with other causes, diminishes the activity of the cell. Indeed, the polarization may become so great as to cause the E.M.F. to fall to zero. A chemical, called the depolarizer, is introduced to remove the hydrogen, or to prevent its being formed. Cells intended for open circuit work contain a depolarizing agent that acts very slowly; thus they polarize rapidly if left on closed circuit, but recover if left for a time on open circuit. Cells intended for closed circuit work should polarize very little, thus the depolarizing agent should act quickly. The deterioration of a cell, when left on open circuit, due to local action within the cell, is important, but can best be found by actual use, since it takes too long to test this in the laboratory. We might also run an efficiency test by working the cell to exhaustion; but this, too, is better found by actual use. What we are interested in, however, is the behavior of the cell when run on a closed circuit for a given time as the value of a cell is determined

by the rate of its polarization and recovery as well as by its E.M.F. and internal resistance.

49. Experiment 5. *Battery Test.*[1]—Study in this experiment the time variation of: (*a*) Total E.M.F. on open circuit, (*b*) the terminal potential difference on closed circuit, (*c*) the internal resistance, (*d*) the current, and (*e*) the rate of recovery from polarization. The set-up is the same as in Exp. 4 and all quantities are to be measured by the methods there outlined. The difference here is that the key K_1 is left closed all the time except for an instant when it is opened to charge the condenser for measuring the total E.M.F. As above, obtain the readings for the E.M.F. and terminal potential difference for the initial condition of the cell. Now close K_1, and at the end of a minute, charge the condenser and discharge it through the galvanometer, thus obtaining the value of the terminal potential difference on closed circuit after the cell has been delivering current for one minute. As soon as this is done, open K_1 for an instant and charge the condenser; then set K_2 on "insulate" and again close K_1. This key, K_1, must be opened and closed quickly, also these two readings, i.e., for total E.M.F. and terminal P.D., taken as nearly simultaneously as possible. As soon as the galvanometer comes to rest, discharge the condenser through it, obtaining a measure of the E.M.F. on open circuit after the cell had been furnishing current for one minute. This will give a measure of the polarization. Repeat these readings every minute for five minutes and then every five minutes for twenty-five minutes more. At the end of this time, open K_1 and measure the E.M.F. of the cell as it recovers for another thirty minutes. As above, take readings at first every minute and then at intervals of five minutes. Find out from the instructor what resistance to use for R. Read Exp. 4 before attempting this one. Practice with another cell as there suggested.

Report.—1. Compute the total E.M.F. and terminal potential difference in volts, and internal resistance in ohms.

2. Plot on one sheet, with time as abscissas, the total E.M.F., the terminal potential difference, the internal resistance, and the current as ordinates.

3. Plot also on the same sheet the recovery curve, starting at the other end of the time axis, running the curve backwards.

[1] CARHART, Primary Batteries.

CHAPTER IV

MEASUREMENT OF POTENTIAL DIFFERENCE

50. Description of a Potentiometer.[1]—There is, perhaps, no single electrical instrument which has so wide a field of usefulness and which gives, at the same time, such trustworthy results as the potentiometer. While comparing potentials primarily, it may, with proper accessories, be adapted to compare currents and resistances as well, and is so easy to manipulate as to be an effective instrument even in the hands of a novice. The funda-

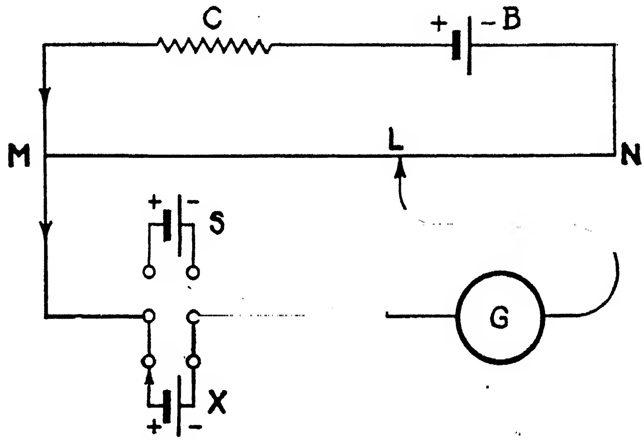

FIG. 25.—Simple Potentiometer circuit.

mental principle of the potentiometer may be illustrated by Fig. 25, where MN is a wire of uniform resistance, stretched along a scale with equal divisions and supplied with current from a battery B, whose E.M.F. must be larger than those to be compared. If the polarity of B is as shown, M will be at a higher potential than N, and the fall of potential per unit length will be the same all along the wire. If the difference of potential between M and N is known, the wire may be regarded as a potential measur-

[1] LAWS, Electrical Measurements, p. 271.
Electrical Meterman's Handbook, p. 208.
KARAPETOFF, Experimental Engineering, p. 74.

ing rod. To measure an unknown E.M.F., such as the battery X of the figure, an auxiliary circuit MXL is provided, containing a galvanometer and key. If the battery X were temporarily removed and a short circuiting wire substituded in its place, a portion of the current from B would flow in the shunt circuit from M to L, causing a deflection of the galvanometer in a particular direction. If, instead, the battery B were removed, X would cause a current to flow in the direction $XMLG$, giving a reverse deflection. If, however, both batteries are included, and the slider L is adjusted until the IR drop in the wire due to the current from B is exactly equal to the E.M.F. of X, no current will flow in the shunt, indicated by zero deflection of the galvanometer. The current in the circuit BNM is then just the same as though the shunt were disconnected. If the potential drop per unit length of the slide wire is known, X may be directly determined, for we have

$$X = \rho I l_1, \tag{1}$$

where ρ is the resistance per unit length, and l_1 the length required for balance. ρI may be determined by substituting for X a cell S of known E.M.F. and balancing as before. Let l_2 be the length required for this balance. Then

$$S = \rho I l_2 \tag{2}$$

Whence

$$\rho I = \frac{S}{l_2} \tag{3}$$

and

$$X = S\frac{l_1}{l_2} \tag{4}$$

The unknown E.M.F. is thus obtained in terms of S by a direct comparison of the lengths l_1 and l_2. If the fall of potential per unit length of wire were some decimal fraction of a volt, the unknown X could be read from the slide wire directly, thus avoiding the calculation indicated. The method of accomplishing this may be illustrated by the following example: Suppose the slide wire MN contains 200 divisions, and the fall of potential between M an N is 2 volts. The fall of potential per division is then .01 volt. Let the standard cell have an E.M.F. of 1.0185 volts. Set the slider at 101.85 divisions, include S in the shunt circuit, and obtain a balance, not by moving the slider, but by varying the control resistance C, thereby changing the current I. When a balance has been secured, $\rho I = \dfrac{1.0185}{101.85}$

.01 volt per division. The potentiometer is now standardized. Substitute X for S and balance by moving L, leaving C unchanged. If this reading should be 145.63 divisions, the E.M.F. of the unknown cell would be 1.4563 volts. When used in this manner, the instrument is said to be a "Direct Reading Potentiometer."

To carry out comparisons with great accuracy, a very long wire, having a high degree of uniformity, is obviously necessary. Since such a wire is difficult to obtain, and inconvenient to use, it is customary to substitute for it two resistances, as shown in Fig. 26. If the sum of R_1 and R_2 is kept constant, they together

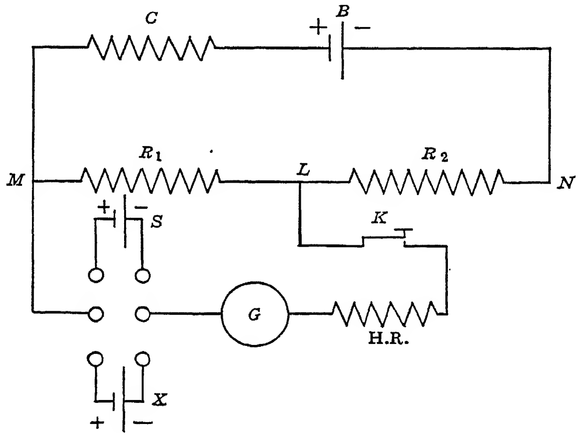

FIG. 26.—Potentiometer constructed from resistance boxes.

are equivalent to a wire of fixed length, and an increase in R_1 accompanied by an equal decrease in R_2 is equivalent to moving the slider of Fig. 25 to the right, while an increase in R_2 and a decrease in R_1 moves it to the left. Balances may easily be obtained, the conditions for which are the same as outlined above. For example, when a balance has been obtained with X in circuit, we have

$$X = R_1 i. \tag{5}$$

where R_1 is the resistance required for balance and i, the current flowing through the potentiometer, i.e., through CR_1R_2. This current, which may be obtained by balancing against the standard cell S, is given by

$$S = R_1'i \text{ or } i = \frac{S}{R'_1} \tag{6}$$

where R'_1 is the value required for balance in this case. Whence,

$$X = \frac{R_1}{R'_1} S \tag{7}$$

It must constantly be borne in mind that the above relations require that i should remain constant during the entire process, which will be true only when the sum of the resistances, $R_1 + R_2 = C$ is unchanged, and the E.M.F. of B is constant. This arrangement may be made "Direct Reading" if i is a known decimal fraction of an ampere. This may be accomplished by giving to R'_1 a value having the same significant figures as the E.M.F. of the standard cell, and balancing by varying C. For example, suppose, as above, $S = 1.0185$ volts and the boxes used have resistances in the neighborhood of 20,000 ohms. If $R'_1 = 10{,}185$ ohms, when a balance has been reached

$$i = \frac{1.0185}{10{,}185} = \frac{1}{10{,}000} \text{ ampere,}$$

and the fall of potential across each ohm is $\frac{1}{10{,}000}$ of a volt. Replacing now S by X and balancing again, leaving C undisturbed and keeping $R_1 + R_2$ constant,

$$X = \frac{R_1}{10{,}000} \tag{8}$$

51. Standard Potentiometers.—In order to avoid the necessity of providing two exactly similar boxes, making the various connections as explained above, and transferring plugs from one to the other, it is convenient to have a single instrument, including all resistances, switches, keys, etc., provided with binding posts, to which the various E.M.F.'s may be connected. A number of such potentiometers are on the market, three of which will be described.

1. *The Leeds and Northrup Potentiometer.*—The arrangement of this circuit, which is the simplest of those to be studied, is shown diagrammatically in Fig. 27, in which the letters correspond, as far as possible, to those used in Fig. 26. The potentiometer circuit proper, $BNMC$, consists of 16 coils of 5 ohms each, and a long slide wire NO, also of 5 ohms. This circuit, in normal operation, carries one fiftieth of an ampere, giving across each coil as well as the slide wire, one-tenth of a volt fall of potential. The circuit, containing the unknown potential is included between the movable contacts, T and L. The box R_1 of the previous

diagram is that part of the circuit lying between T and L, while R_2 consists of two parts, namely, the right-hand portion of the slide wire, and the resistance to the left of T. The slide wire, shown in the figure by a single turn, in reality consists of ten turns wound on a marble cylinder and is about 17 feet in length. The fall of potential across each turn is thus .01 volt, and, as the dial circle is divided into 200 parts, the instrument

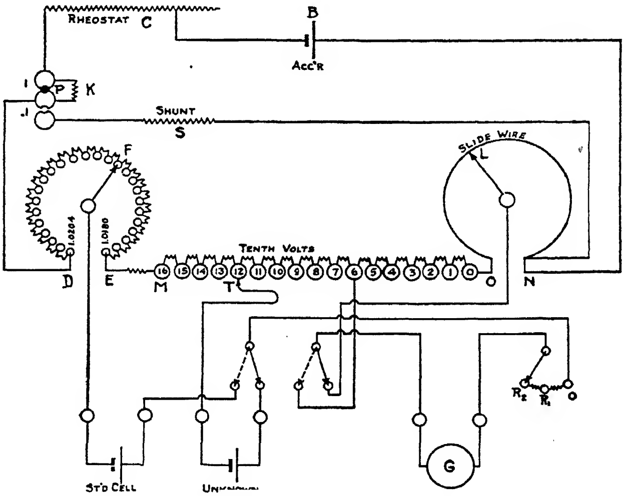

FIG. 27.—Diagram of Leeds and Northrup potentiometer.

reads directly to .00005 volt. By moving the slider L and the contact T, the difference of potential may be varied by infinitesimal changes from 0 to 1.7 volts.

The range of the instrument, for small electromotive forces, such as those furnished by thermo-couples, is increased tenfold by means of a shunt. As seen from the diagram, when the plug P is inserted in the hole marked .1, the shunt S, which contains a resistance of one-ninth that of the potentiometer proper, is connected across the entire circuit, so that only one-tenth of the normal current flows through the potentiometer proper. In order that the current from the battery B may remain unchanged a resistance K is automatically included, thus keeping the resistance of the entire circuit the same. A ready means of standardizing the potentiometer current is furnished by the extra dial DE, containing 19 coils of such a resistance, that, with the

normal current flowing, the fall of potential across each is .0001 volt. From the .6 post of the tenth volt dial, a permanent lead is brought out, and connected, when the selecting switches are thrown toward the left, through the galvanometer and standard cell to the contact F. The fall of potential from the .6 plug to M is 1 volt; from M through the resistance to E is .018 volt, and from E to F as many ten-thousandths of a volt additional as may be required to equal the E.M.F. of any normal Weston cell within the ordinary range of temperature (1.0180–1.0204 volts).

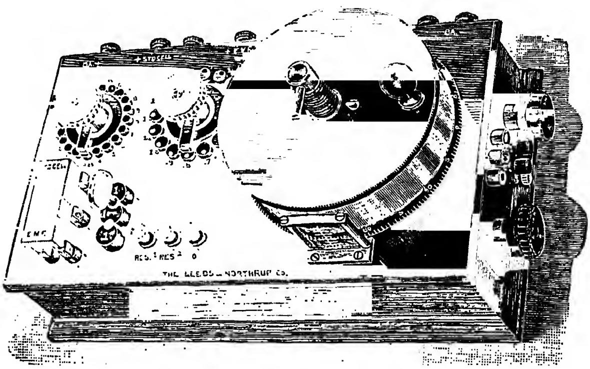

Fig. 28.—Leeds and Northrup potentiometer.

It is thus possible to check the potentiometer current without re-setting the instrument. The operation then is as follows: Set the standard cell dial to correspond to the E.M.F. of the cell, corrected for temperature. Move the selecting switch to the left, set P in the hole marked 1, and vary the control resistance C (usually mounted at the right-hand end of the instrument) until a balance is obtained. One-fiftieth of an ampere, the normal current is now flowing. Move the selecting switch to the right, thus including the unknown E.M.F., and vary T and L until the balance is once more obtained, when the unknown may be read directly. If it is less than .15 volt, set P in the .1 hole, and balance again, when the reading of the instrument must be divided by 10. The complete instrument is shown in Fig. 28.

2. *The Wolff Potentiometer.*—The fundamental principle of this instrument is shown by the simplified connections of Fig. 29. The result which must be secured by any arrangement is

that the resistance of the potentiometer circuit proper, namely, *MN*, must be kept constant, while the resistance across which the auxiliary circuit *FL* is connected, must be continuously variable. By moving *F* and *L*, changes of 1,000 and 100 ohms, respectively, are obtained, without changing the total resistance as is at once obvious. The resistance coils connected by the double sliders are sets with units of 10, 1, and .1 ohms respectively. These double sliders are mounted so as to move together, but are insulated from each other and connected in circuit as

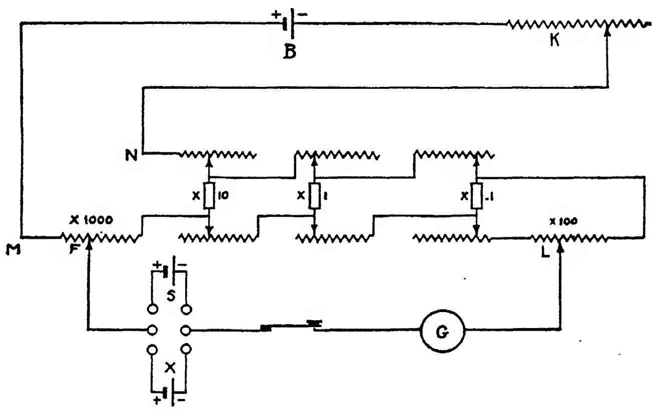

FIG. 29.—Principle of Wolff potentiometer.

shown in the diagram. If the pair at the left is moved one division to the right, it is evident that the resistance between *N* and *L* is increased by 10 ohms, while that between *F* and *L* is decreased by the same amount, thus leaving the resistance between *M* and *N* unchanged. In the same way, the middle pair of sliders produces changes of 1 ohm each between *F* and *L* leaving *MN* unchanged, while the right-hand pair produces changes of .1 ohm each. Shifting any one of these sliders, therefore, is equivalent to moving the slider *L* of Fig. 25 by definite amounts.

The actual wiring of the instrument, mounting of the sliding contacts, connections to accessories, switches, etc., are shown in Fig. 30. The control resistance *K* is not included in the instrument. Any ordinary resistance box capable of small variations will serve for this purpose. The total resistance of the instrument, as sketched, is 19,000 ohms. When carrying the normal current of one ten-thousandth of an ampere, the difference of

potential across consecutive posts of the first dial is one-tenth of a volt; of the second dial, one-hundredth of a volt; and of the last dial, one hundred-thousandth of a volt, while the maximum voltage directly measurable is one and nine-tenths volts.

In using this instrument, first obtain the E.M.F. of the standard cell, corrected for temperature, and set the small middle dial of the upper row at this value. Set the switch in the upper left-hand corner at *NN*, and the one in the right-hand corner, which includes a high resistance in the galvanometer circuit, at its largest value. Obtain a balance by varying the control resis-

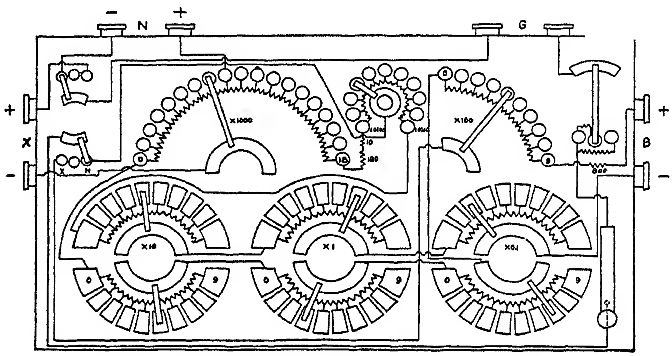

FIG. 30.—Wiring diagram for Wolff potentiometer.

tance, cutting out the galvanometer resistance as a balance is approached. This operation standardizes the current at one ten-thousandth of an ampere. Now switch to *XX* and balance by setting the large dials, when the unknown may be read off directly. In checking the potentiometer current, which must frequently be done, it is not necessary to change the dials from their positions when balanced on the unknown E.M.F.

3. *The Tinsley Potentiometer.*—The working diagram for this instrument, which is unique in that it employs an electrical vernier, is shown in Fig. 31. Seventeen coils, with a resistance of 5 ohms each, connected in series with a short slide wire of .5 ohm, form the potentiometer circuit proper *MN*, while the auxiliary circuit is *FGL*. Attached to a movable arm are two sliding contacts, so spaced that they always rest upon two alternate posts, leaving one post between them as indicated. This pair of contacts is connected to a second series of 10 coils of 1 ohm each.

When the normal current of one-fiftieth of an ampere is flowing through MN, the fall of potential between adjacent posts is .1 volt. However, the fall of potential between the posts connected by the pair of contacts to the second series of coils is also .1 volt, since the two coils of the main circuit are now shunted by a resistance equal to their own, giving a resultant resistance between the contacts equal to that of a single main circuit coil. Between adjacent posts of the second series there is accordingly .01 volt fall of potential, and across the slide wire there is also .01 volt potential difference. This instrument, like the Leeds and Northrup, is provided with separate connections for the standard cell,

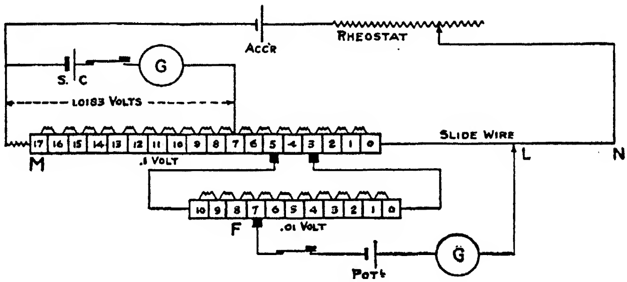

FIG. 31.—Principle of Tinsley potentiometer.

so that it is not necessary to re-set all of the sliders when checking the current through the potentiometer circuit proper. A standard cell lead is permanently attached to post number 7. Across the ten coils between it and post 17 there is, accordingly, 1 volt potential difference, and in series with the main circuit is another coil shown at the left of 17, of such a value that, with the normal current flowing, the fall of potential across it is .0183 volts, and to the other side of this, the second standard cell terminal is attached. Unlike the Leeds and Northrup instrument, this coil cannot be varied to compensate for variations in the E.M.F. of the standard cell due to temperature changes, but the value 1.0183 volts is sufficiently accurate for ordinary purposes.

The wiring connections, switches, etc., are shown in Fig. 32. The control rheostat is included in the instrument, and consists of the dial in the right-hand corner and the slide wire immediately above it. By moving the plug in the upper left-hand corner to the hole marked X by .1, the instrument is shunted by a resistance of such a value that all readings should be divided by ten, a

feature of great importance in thermo-couple work. In using the instrument, set the shunt plug in the hole marked X by 1, and the two-point switch below the middle dial on SC. The first dial must be set so as to shunt none of the coils above the seventh, otherwise, the resistance over which the standard cell is to be balanced will be reduced effectively by one coil. A good rule is to set this dial always at zero when balancing on the standard cell. Obtain a balance by changing first the rheostat and then the slider above it, which is provided with a slow motion screw for the final setting. Use in this connection the key marked

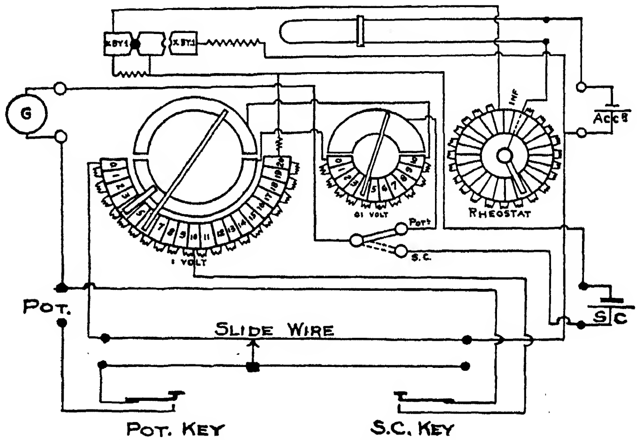

FIG. 32.—Wiring diagram of Tinsley potentiometer.

SC. The current is now exactly one-fiftieth of an ampere and the instrument is standardized. To measure the unknown, simply move the two-point switch to Potl., and balance by setting the two dials and the lower slide wire. If the unknown is less than .2 volt, use the shunt as explained above, dividing the reading by 10. In carrying out any measurement, the current through the instrument should be checked frequently.

52. The Weston Standard Cell.—While the legal definition of electromotive force is given in terms of the standard current and resistance by means of Ohm's law, nevertheless, in actual practice, the volt is specified in terms of the standard cell. After many years of investigation, the Weston standard cell has been perfected to such an extent that persons in different parts of the

world, may, by following definite specifications, construct cells of this type and be sure of securing E.M.F.'s which agree within less than 1 part in 10,000. This cell is usually set up in an air-tight H-shaped vessel, as shown in Fig. 33, with platinum wires sealed through the bottoms for connection with the electrodes. The positive electrode consists of pure mercury while the negative is an amalgam of cadmium and mercury. These are placed in the bottoms of the tubes, and a solution of $CdSO_4$ with a few extra crystals to insure saturation, forms the electrolyte between them. To protect the mercury against contamination by the $CdSO_4$ and at the same time prevent polarization, a thick paste, consisting mainly of mercurous sulphate, is placed over the mercury. As the cell operates, the cadmium ions from the $CdSO_4$ solution displace some of the ions from the mercurous sulphate paste and mercury is deposited upon the mercury electrode.

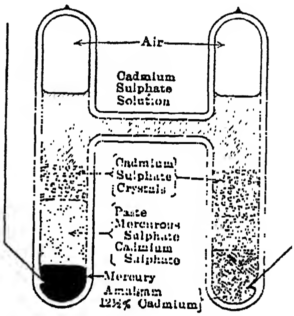

FIG. 33.—Weston standard cell.

One of the advantages of this cell over former types is that its electromotive force changes but very little with the temperature. The electromotive force of a cell which has been set up with care is given, with accuracy sufficient for most purposes, by the equation

$$E_t = E_{20} - 0.0000406\,.(t - 20°\ \mathrm{C.}). \qquad (9)$$

That is, the E.M.F. decreases 0.0000406 volt for each degree the temperature is raised above 20° C., and increases by the same amount for each degree below 20° C. This quantity is called the temperature coefficient. Since standard cells are never used as a source of current, but merely for balancing potentials or charging condensers, they are made of small size. Those furnished in the laboratory are mounted in a brass tube, with a hard rubber top provided with binding posts and a hole through which to insert a thermometer. The E.M.F. of the individual cells is usually given at 20° C., from which the E.M.F. at the temperature at which they are used may be computed by means of the formula given above. When used in a potentiometer circuit, a high resistance should be included and gradually cut out as a balance is approached.

53. Experiment 6. *Comparison of Cells by the Potentiometer. A. Simple Potentiometer.*—Connect the apparatus, as shown in Fig. 26, Art. 50, omitting the control resistance C, and using for R_1 and R_2 two exactly similar boxes. B should be a cell of constant E.M.F., preferably a portable storage battery. Obtain from the instructor a standard and several unknown cells whose E.M.F.'s are to be determined. The high resistance marked H.R. need not be known accurately, since its purpose is merely to protect the galvanometer and standard cell from excessive currents when the potentiometer is far from balance. It is well to start this at about 10,000 ohms, gradually reducing it as a balance is approached. Be sure that the double pole double throw switch for connecting S and X in circuit is not provided with cross wires, as they would short circuit the cells. To keep $R_1 + R_2$ constant, as required in the theory, start with all the plugs out of R_1 and all in R_2, and obtain a balance by transferring them from their places in one box to the corresponding holes in the other. $R_1 + R_2$ will then always remain equal to the total resistance of one box. To test whether the polarity of the cells is properly arranged in the two circuits, first rock the double pole double throw switch on X, break the potentiometer circuit at B, tap the key K lightly, and note the direction of swing of the galvanometer. Now close again the circuit at B, remove the wires from the middle posts of the double pole double throw switch, and join them. The galvanometer should swing in the opposite direction on tapping the key. First, secure a balance on X; then rock the switch over and balance on S, afterwards checking your balance on X, to make sure that the potentiometer current has not changed during the process. Reverse the connections at B, also on the auxiliary circuit, and proceed as before, taking the average of the two results thus obtained. This is necessary to eliminate errors due to spurious contact and thermal E.M.F.'s within the potentiometer.

B. Direct Reading Potentiometer.—Include in the potentiometer circuit the control resistance C, as shown in Fig. 26. Determine the temperature of the standard cell and its E.M.F. corrected for this temperature. Set R_1 to have the same significant figures as this E.M.F., using the largest multiple possible, and put R_2 equal to the difference between the total capacity of one box and R_1. Switch S into the shunt circuit and balance by varying C. Then rock over on to X and, leaving C fixed, balance

by plugging back and forth between R_1 and R_2, keeping their sum constant. The reading of R_1, when properly pointed off, gives X directly. After each balance on X, the setting on the standard cell should be checked and C changed, if the current has not remained constant, which, of course, necessitates a new balance on X. Now reverse terminals as in Part A, and repeat, taking the average of the two results.

Report.—1. Give values of E.M.F. for all cells compared, and where temperature corrections are known, reduce to 20° C.

2. Suppose a balance has been obtained without H.R. in circuit. Now include H.R. How will the balance point be affected? Why?

3. What is the maximum E.M.F. that may be measured by the direct reading potentiometer, as you have used it in this experiment?

54. The Volt Box.—In standard potentiometers, operated on normal current, the maximum difference of potential which may

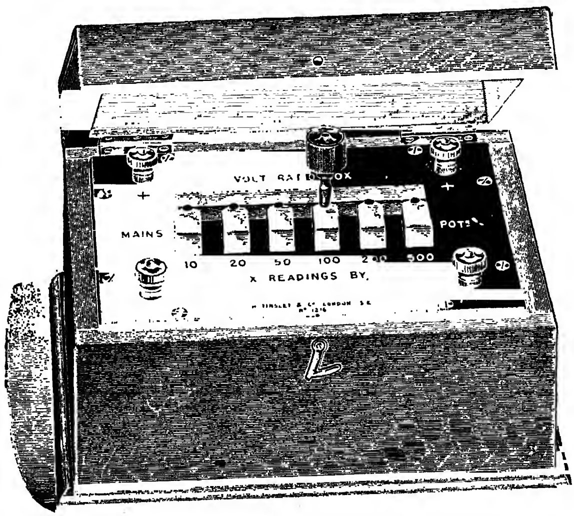

Fig. 34.—Volt box.

be measured directly never exceeds two volts and is usually even less. When it is desired to measure voltages in excess of this value, some means must be provided for accurately dividing the unknown voltage into definite fractions of the total, small enough to be measured by the potentiometer available. This may be

accomplished by means of the "volt box." This consists of an accurately adjusted resistance box, of large range, in which the blocks to which the coils are attached are provided with sockets for receiving traveling plugs. The voltage to be divided is impressed across the terminals and the fraction to be measured is obtained across the traveling plugs, which may be set at any points desired. By Ohm's law, the voltage across the traveling plugs is such a fraction of the total voltage as the resistance between the traveling plugs is of the total resistance. If the resistance of the volt box is 10,000 ohms, the drop across 1,000 ohms is one-tenth of the total; that across 100 ohms, one-hundredth of the total, and so on. It is simpler to use decimal ratios wherever practicable. Special boxes are made in which these ratios are obtained by setting a dial switch or a single plug as shown in Fig. 34.

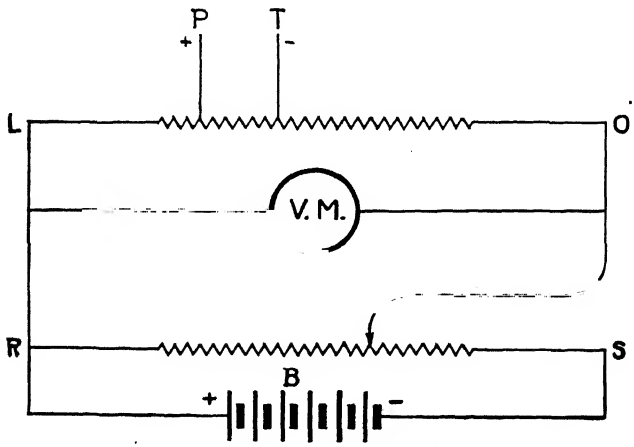

FIG. 35.—Connections for standardizing a volt meter.

55. Experiment 7. *Calibration of a voltmeter by Potentiometer and Volt Box.*[1]—The method, in brief, consists in impressing across the terminals of the voltmeter various voltages and measuring these voltages by means of a potentiometer provided with a volt box. The connections for this purpose are shown in Fig. 35. *VM* is the voltmeter to be calibrated; *LO* the volt box; *RS* a high resistance rheostat with a sliding contact for

[1] JANSKY, Electrical Meters, chap. V.
KARAPETOFF, Experimental Engineering, vol. I, pp. 51–55.

voltage regulation, and B a storage battery. P and T are the terminals from the traveling plugs of the volt box which are to be attached to the potentiometer. The voltage of B should be sufficient to give full scale deflection of the instrument. Use any one of the potentiometers described above, following the directions given for each instrument. After the connections with the potentiometer have been properly made and its current adjusted by balancing against the standard cell, throw the selecting switch to the point marked "unknown." Set the slider of the rheostat RS so that the voltmeter indicates about one-tenth full scale deflection, and choose the largest. decimal ratio of the volt box giving a voltage within the range of the potentiometer. Measure this voltage with the potentiometer. In a similar manner check the voltmeter at 8 or 10 points distributed uniformly across the scale. Test the constancy of the potentiometer current frequently by re-balancing against the standard cell. Record voltmeter readings, potentiometer settings, and volt box ratios. Note carefully the zero reading of the voltmeter before beginning the test and again at the end, after it has been deflected for some time, to see if the springs show any elastic fatigue. With about two-thirds full scale deflection, place the instrument in a vertical position to test the accuracy with which the moving system is balanced. Bring another instrument near this one, and see if there is any effect from external magnetic fields. Tap the instrument gently with the finger to see if the bearing friction is large. Does the pointer swing past its final position when a voltage is suddenly thrown on?

Report.—1. Obtain the differences between the readings of the instrument and true voltages, and plot these corrections as ordinates against readings of the instrument as abscissas. Draw in straight lines connecting these points.

2. State your findings regarding the imperfections of the instrument.

3. Would it indicate on alternating voltages?

CHAPTER V

MEASUREMENT OF CURRENT

56. Kelvin's Balance.—This is an instrument for the measurement of current in which use is made, not of the action between the magnetic field of a current and a permanent magnet, as in the case of galvanometers and ammeters, but of the action between the fields of two currents. It consists of six flat coils placed horizontally, four of which are fixed while the other two, mounted at the ends of a beam pivoted at the middle, are movable. The general arrangement is shown in Fig. 36. The current to be measured passes through all six coils in series, flowing in each in

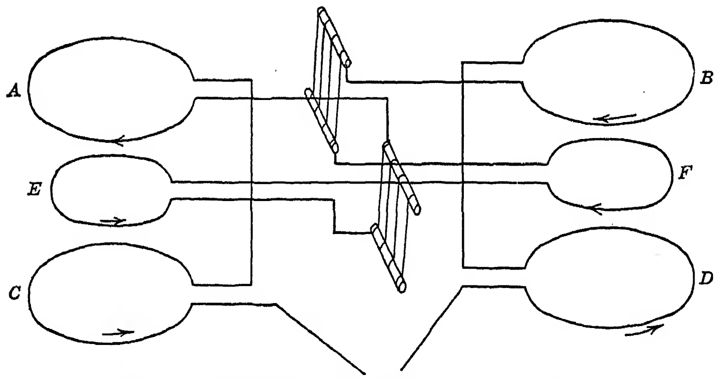

FIG. 36.—Arrangement of coils in Kelvin's balance.

such a direction that A and C both urge E downward, while B and D urge F upward. The force of attraction or repulsion between two coils is proportional to the current in each. Accordingly, when the coils are connected in series the force between them is proportional to the square of the current. Thus, the electrodynamic action between the fixed and the movable coils is such as to produce a torque on the movable ones in the counter clockwise direction proportional to the square of the current. This torque is counterbalanced by a weight which slides along a graduated beam attached to the moving system. An index at each end shows when a balance has been reached. Since the

torque due to the current is proportional to the square of the current, and that due to the weight is proportional to the weight and the length of the lever arm, we have, as the condition for equilibrium,

$$KI^2 = WL \tag{1}$$

where W is the weight of the slider, L its distance from the zero position, and K a constant depending upon the construction of the instrument. Solving

$$I^2 = \frac{W}{K}L \tag{2}$$

or

$$I = \text{const.}\sqrt{L} \tag{3}$$

The constant is generally so given that one must use the doubled square root of the length L, and, to facilitate observations, tables of these quantities have been prepared. For rough work, however, a fixed scale is mounted directly behind and a little above the movable one, from which the doubled square root may, with fair approximation, be read directly. Since the constant depends upon the weight of the slider, a means is afforded for changing the range. Four weights are usually supplied for which the constants are 0.025, 0.05, 0.1, and 0.2, giving ranges of 1.25, 2.5, 5, and 10 amperes, respectively, since the movable scale has 625 divisions, giving a doubled square root of 50.

As with an ordinary balance, the beam must be in equilibrium for no load, that is, no current flowing through the coils. If the index at the end does not read zero, equilibrium may be obtained by moving a small metal flag attached to the moving system so as to throw more of its weight to one side or the other, as is required. A special device mounted on the base and operated by a handle below the case is provided for this purpose. The movable system is carried by flexible ligaments made up of a number of fine phosphor-bronze ribbons placed side by side. As these are delicate and easily broken, an arrestment is provided which is operated by a milled head at the bottom of the case. Weights should never be changed without first raising the arrestment. Since the balance must be in equilibrium for zero current, no matter which weight is used, there must be a separate counterpoise for each. These consist of brass cylinders, provided with a pin, which are placed in a small horizontal trough at the right-hand end of the moving system, with one end of the pin

passing through the hole in the bottom of the trough. Since the direction of the torque is independent of the direction of the

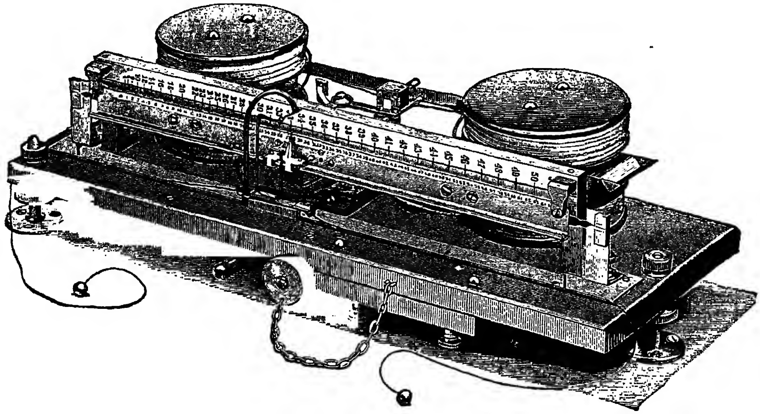

FIG. 37.—Kelvin's balance.

current, the instrument may be used either on direct or alternating currents, indicating in the latter case, root mean square values. Figure 37 shows the usual laboratory form of the Kelvin's balance.

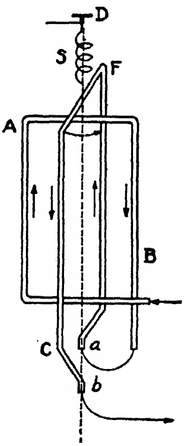

FIG. 38.—Arrangement of coils in Siemens electrodynamometer.

57. The Siemens Electrodynamometer.—This is another current measuring instrument working on the principle of the electrodynamic action between two coils carrying currents. The coils are rectangular in form and placed perpendicular to one another, as shown in Fig. 38. The movable coil, *CF*, which is placed outside the fixed coil *AB*, is carried by a fine point resting in a jewel and the current is led to and from it by wires dipping into mercury cups at *a* and *b*, situated one above the other in the axis of rotation. One end of a helical spring *S* is attached to the moving coil, while the other is fastened to a milled head *D* carrying an index read from a fixed circular scale. When a current flows through the two coils in series, the movable one tends to set itself parallel to the fixed, but is brought back to its zero position by turning the head *D*, thus

twisting the spring. The torque due to the current is proportional to the square of the current since the coils are in series, while that due to the spring, by Hooke's Law, is proportional to the angle through which it is twisted. Accordingly, we have, as the condition for equilibrium,

$$I^2 = A^2\phi$$

or

$$I = A\sqrt{\phi}$$

where A is a constant depending upon the size of the coils, number of turns, stiffness of spring, etc, and ϕ, the angle through which the spring is twisted. The range of the instrument is changed by varying the number of turns in one of the coils. The instrument usually has two fixed coils with separate binding posts on the base. Since the magnetic field of these coils is small, that of the earth is appreciable in comparison and may introduce an error. For example, if the earth's field is in the same direction as that of the fixed coil, the instrument will read too high, while if the earth's field is opposite, it will read too low. This error may be eliminated by reversing the currents and taking the average. Since the direction of rotation of the movable coil is independent of the direction of the current, the instrument will indicate on alternating currents as well as direct, giving in the latter case, root mean square values. It may accordingly be calibrated on direct and used on either direct or alternating currents. Figure 39 shows the usual form of Siemens electrodynamometer.

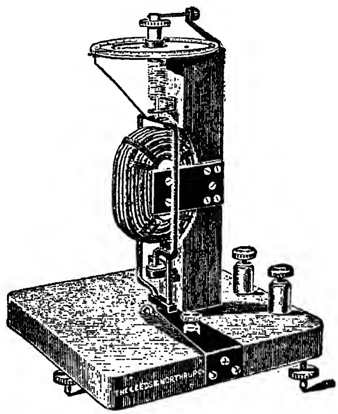

FIG. 39.—Siemens electrodynamometer.

58. Experiment 8. *Calibration of an Electrodynamometer.*[1]—In this experiment, an electrodynamometer is to be calibrated in terms of a Kelvin balance, which is taken as the standard instrument. Connect the instruments in series on a 20-volt

[1] JANSKY, Electrical Meters, chap. VIII.
CARHART and PATTERSON, Electrical Measurements, chap. III.

storage battery, including a variable rheostat and an ammeter to observe roughly the currents used. Both instruments must first be leveled and adjusted for zero on no current. Begin with the lowest range of the Kelvin balance. For this use the carriage alone and the smallest counter weight. When the limit of this range has been reached, raise the arrestment, open the case, and push the carriage moving mechanism a little to one side bringing it forward enough for clearance. Place the first additional weight upon the carriage, and the second counter-poise in the trough. Whenever new weights are put in position, the zero must be rechecked. Measure in this way the currents for ten points on the electrodynamometer, taking them a little closer at the lower end of the scale. Record electrodynamometer readings, Kelvin balance readings, and number of counterpoise.

Report.—1. Compute the current for each setting of the instrument, also the constant A in equation (5).

2. Plot current as ordinates and settings as abscissas. What is the shape of this curve?

3. What is meant by the root mean square value of an alternating current?

4. Name some other electrical instruments operating on the principle of the electrodynamometer.

59. Ammeters and Voltmeters.[1]—An ammeter, as the name implies, is an instrument for measuring the current flowing in a circuit; while a voltmeter measures the difference of potential or

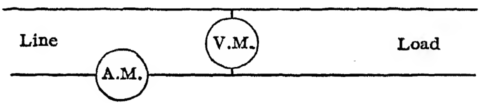

FIG. 40.—Connections for ammeter and voltmeter.

electrical pressure existing between two points Since the former indicates, at any instant, the rate of flow of electricity through a conductor, it must be placed in series with the circuit, so as to be traversed by the entire current; while the latter, being a pressure gauge, is connected in parallel with the circuit, and carries a very small current, which in general may be neglected. The regular method of connecting these instruments is shown in Fig. 40.

[1] JANSKY, Electrical Meters, chap. III.
KARAPETOFF, Experimental Electrical Engineering, vol. I, chap. II.
Electrical Meterman's Handbook, chap. V.

While many different kinds of indicating instruments are in use, each having its particular field of application, those generally employed in direct current work are of the "moving coil" type, and are the only ones which will be considered here. The working parts of instruments of this class are the same in both voltmeters and ammeters, the differences between them being only in the method of connection. The instrument proper is, in reality, a low sensibility, portable D'Arsonval galvanometer, consisting of a coil of fine wire, well-balanced, and pivoted between the poles

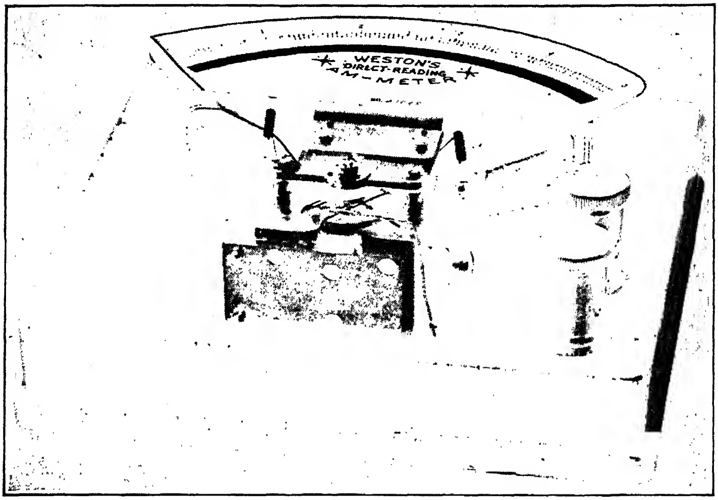

FIG. 41.—Working parts of Weston ammeter.

of a strong, permanent horse-shoe magnet. The magnetic flux through the coil is increased by placing within it a cylindrical iron core, while the air gap is further reduced by pole tips shaped in such a manner as to make the field as nearly radial as possible, with respect to the axis of the coil. In this way, the torque acting upon the coil, when traversed by a current, is independent of its angular position, the condition necessary for equal scale divisions. The current is led to and from the coil by spiral springs, which furnish also the opposing torque. The current sensibility of such an instrument is such that a few thousandths of an ampere, or less, will give a full scale deflection; and since the resistance of the instrument is low, a few millivolts across its terminals will furnish

this current. Figure 41 shows the construction of a Weston ammeter.

When it is desired to construct an ammeter, the instrument G is provided with a shunt, S, as shown in Fig. 42. The shunt, which carries the current to be measured, has a resistance (always low) such that it gives, when carrying the maximum current for which it is designed, a fall of potential across its terminals equal to that required for full scale deflection of the instrument. For example, suppose 50 millivolts are required for full scale deflection, and an ammeter reading to 25 amperes is desired; the resistance of the shunt must be

$$R = \frac{.050}{25} = .002 \text{ ohms}$$

By the law of shunts, the current through the instrument (neglected in the above calculation) is proportional to that through the shunt; and if the scale is divided into 25 equal parts, we have an ammeter of the desired range.

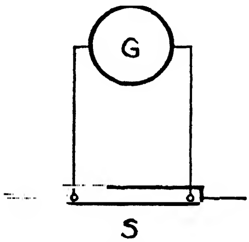

Fig. 42.—Internal connections for ammeter.

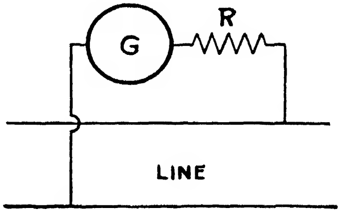

Fig. 43.—Internal connections for voltmeter.

The same instrument may be used as a voltmeter, if, instead of the shunt, it is connected in series with a large resistance R, Fig. 43, of such a value that, when the maximum voltage to be measured is impressed across the outside terminals of G and R, the drop across the instrument is that required for full scale deflection. For example, suppose the instrument, as above, requires 50 millivolts for full scale deflection, that it has a resistance of 10 ohms, and that it is desired to construct a voltmeter reading to 100 volts. By Ohm's law, R, is given by the following equation:

$$\frac{.050}{99.95} = \frac{10}{R}$$

Whence

$$R = \frac{10 \times 99.95}{.05} = 19{,}990 \text{ ohms}$$

Since the current through the instrument is proportional to the external voltage impressed, if the scale is divided into 100 equal parts, we have the voltmeter required. In some instruments, e.g., Weston, especially for low ranges, the shunts and series resistances, or multipliers, as they are generally called, are placed within the case and cannot be seen; while in others, e.g., Siemens and Halske, and R. W. Paul, they are mounted outside the case and are detachable. The latter have the advantage of being interchangeable, so that the same instrument, when provided with a series of shunts and multipliers of appropriate values, may serve either as a voltmeter or as an ammeter with any number of ranges for each.

60. Experiment 9. *Electrical Adjustment of an Ammeter and a Voltmeter.*—It is the purpose of this exercise to illustrate the fundamental principles of construction and operation of moving coil ammeters and voltmeters. For this purpose, a Weston switch-board type instrument, with transparent case, has been provided with an adjustable external shunt and series resistance. It is to be standardized and tested, first as an ammeter, and then as a voltmeter. In order to accomplish this, it is necessary to know three things concerning the instrument: (1) Resistance; (2) current sensibility; (3) millivolts for full scale deflection.

1. The resistance of the instrument may be obtained directly by means of a Wheatstone bridge. Set the ratio coils Fig. 15 with 100 ohms in the right-hand bank and 10,000 in the left. Be careful to connect the instrument so that the pointer moves forward when operating the bridge.

2. To find the current sensibility of the instrument, which is defined as the current for unit scale deflection, connect it in series with an adjustable known resistance and a cell whose E.M.F. has been determined. In all the tests to be carried out, remember that the instrument is very sensitive, requiring but an exceedingly small current for full scale deflection. Accordingly, a resistance of at least 1,000 ohms should be included before the circuit is closed. Determine the resistances corresponding to five different indications of the instrument distributed uniformly across the scale, and by Ohm's law, compute the current for unit deflection. The E.M.F. of the cell may be obtained by means of a low range voltmeter.

3. The voltage for full scale deflection is given at once by Ohm's

law as the product of the resistance, the current sensibility, and the number of scale divisions.

Part I. Ammeter.—It is required to construct an ammeter of range 0–5 amperes, from the instrument and adjustable shunt. From Ohm's Law, find the resistance, which, when carrying 5 amperes, gives a potential drop across its terminals equal to the voltage required for full scale deflection of the instrument. Measure the total resistance of the adjustable shunt by means of the bridge used above, correcting for the leads, and find what length of wire is necessary for the required shunt resistance.

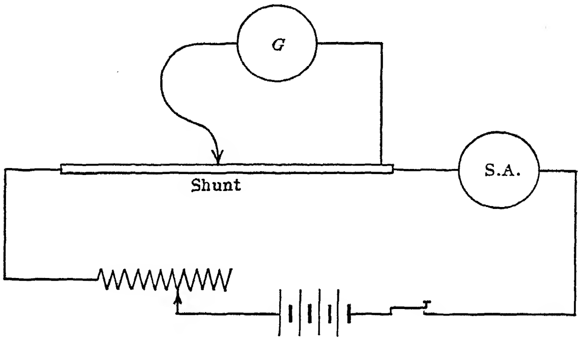

FIG. 44.—Connections for testing improvised ammeter.

Now connect the instrument, as shown in Fig. 44, where *SA* is a standard ammeter and *B*, a storage battery of 6 volts, setting the shunt at the computed value. Check your ammeter against the standard ammeter at 8 or 10 points uniformly distributed across the scale. Now compute, as above, the shunt resistance required in order that your ammeter may have a range of 0–2.5 amperes, and test it in the same manner.

Part II. Voltmeter.—It is required to construct a voltmeter of range 0–50 volts, from the instrument and an adjustable series resistance used as a multiplier. From Ohm's law, compute the resistance which, when placed in series with the instrument, will give the potential drop across it necessary for full scale deflection, when 50 volts are impressed across the instrument and multiplier. Connect the apparatus, as shown in Fig. 45, placing in *M* the computed resistance. *B* is a storage battery of 50 volts, *SV* a standard voltmeter, and *PD* a potential dividing rheostat of several hundred ohms, by means of which

any voltage between 0 and 50 may be impressed across the instruments. Check your voltmeter against the standard at 8 or 10 points evenly distributed across the scale.

Report.—1. Make a sketch of the instrument describing in detail the essential working parts.

2. Outline the general principles involved in adapting it to measure currents and potential differences.

3. Give in full your data and computations for shunts and multiplying resistances.

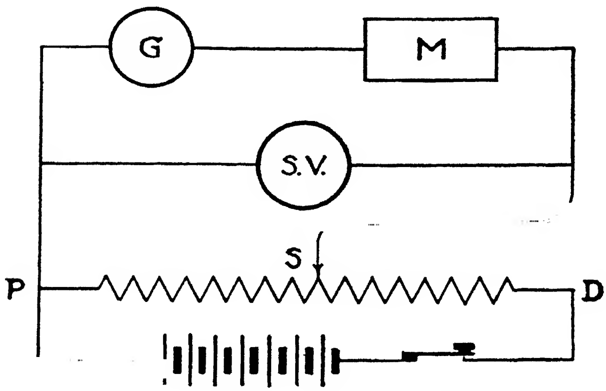

FIG. 45.—Connections for testing improvised voltmeter

4. Give data and curves for your ammeter and voltmeter calibrations.

5. In calculating the resistance of the shunt, in Part I, the current through the instrument was neglected. Compute the error thus made.

61. Measurement of Current by the Potentiometer.—Since the potentiometer measures potentials only, current measurements made by it must necessarily be indirect. For this purpose, use is made of a carefully standardized resistance capable of carrying the current to be measured without appreciable heating. The potentiometer measures the fall of potential across its terminals produced by the current, which is then determined by Ohm's law. If the resistance has some decimal value, the value of the current will have the same significant figures as the potential drop across it. Accordingly, if the potentiometer has been made direct reading for voltage, it will indicate currents directly also.

Resistances for this purpose must be provided with two pairs of binding posts, one for current and the other, for potential. The potential leads are soldered securely to the posts between the current terminals and the effective resistance is only that between the points to which they are attached. Errors from imperfect connections are thus eliminated. Such resistances should be placed in an oil bath to keep the temperature constant. The largest resistance giving, for the desired current, a potential difference within the range of the potentiometer should be used.

62. Experiment 10. *Calibration of an Ammeter by Potentiometer and Standard Resistance.*[1]—Connect the apparatus, as shown in Fig. 46. *AM* is the ammeter to be tested, *B* a storage battery of 10 volts, *S* a rheostat for controlling the current, and

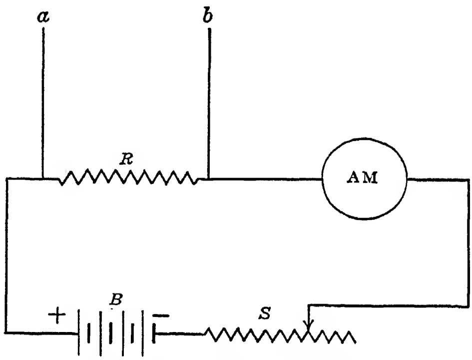

FIG. 46.—Connections for standardizing an ammeter.

R a standard oil-cooled resistance provided with current and potential terminals. The leads *ab* are to be connected to the potentiometer. In connecting up the potentiometer and standardizing the current through it, follow the directions for the particular type of instrument given in Chap. IV. After the potentiometer has been adjusted, cause such a current to flow in the ammeter circuit as will produce about one-tenth full scale deflection and measure the fall of potential across *R* by means of the potentiometer. The resistance *R* and the ammeter carry the same current, since no current flows through *a* and *b* at the point of balance. The current through the ammeter is equal to the

[1] JANSKY, Electrical Meters, chap. V.
KARAPETOFF, Experimental Engineering, vol. I, pp. 51–55.

reading of the potentiometer divided by R. Since R has a decimal value, it is merely a question of properly pointing off this indication. In a similar manner, check the ammeter at 8 or 10 points distributed uniformly across the scale. The balance against the standard cell should frequently be tested and any variations in the potentiometer current compensated.

Record ammeter readings, potentiometer settings, and the value of R. Note carefully the zero reading of the ammeter before beginning the test and again at the end, after the pointer has been deflected for some time, to see if there is any elastic fatigue in the springs. With about two-thirds full scale deflection, place the instrument in a vertical position to test the accuracy with which the moving system is balanced. Bring another instrument near this one to see if there is any effect due to external magnetic fields. Tap the instrument gently with the finger to see if the bearing friction is large. Does the pointer swing past its final indication when a current is suddenly thrown on? Record changes in reading in all of the above cases.

Report.—1. Compute the differences between the readings of the instrument and true amperes.

2. Plot these corrections as ordinates against ammeter readings as abscissas. Draw straight lines connecting these points.

3. State your findings regarding the imperfections of the instrument.

4. Would it indicate on alternating currents?

CHAPTER VI

MEASUREMENT OF POWER

63. Wattmeters.[1]—Whenever a current flows in a circuit, there is a certain amount of energy consumed by the circuit, and any instrument which measures the rate at which energy is consumed is called a wattmeter, from the fact that electrical power is generally measured in watts. Three kinds of wattmeters are in common use; namely, indicating, recording, and integrating. Instruments of the first kind show the power that is being consumed at any instant; those of the second kind make a permanent record on a revolving dial of the power consumption during a given period of time; while those of the third kind show the total energy, that is, the integral of the power times the time, delivered to a circuit during a definite period. Instruments of the first kind only will be considered here, and of the various types in use, only one will be discussed, namely, the electrodynamometer type.

64. Use of an Electrodynamometer for the Measurement of Power.—In case a steady current is flowing through a circuit, the power is given by the product of the current and the fall of potential across the circuit, or

$$\text{Watts} = \text{Amperes} \times \text{Volts}$$

The watts may, therefore, be measured by simultaneously reading an ammeter and a voltmeter. If, however, a single instrument can be devised which will give indications proportional to both current and voltage, it will automatically indicate their product, and may be calibrated to read watts directly. In the discussion of the electrodynamometer, it was pointed out that the torque is proportional to the current in both the fixed and movable coils, and, therefore, to their product. Accordingly, if one of the coils can be made to function as an ammeter and the other as a voltmeter, the instrument will be a wattmeter. For this purpose, the fixed coil is made of a few turns of heavy wire and is connected

[1] Jansky, Electrical Meters, chap. X.
Karapetoff, Experimental Engineering, vol. I, chap. IV.
Electrical Meterman's Handbook.

in series with the circuit like an ammeter, while the movable coil is made of a great many turns of fine wire having a high resistance and is connected across the circuit like a voltmeter and carries a current proportional to the voltage. The torque is proportional, therefore, to amperes times volts, hence, to watts. This is the principle underlying the Weston Indicating wattmeter, the connections for which are shown in Fig. 47. A and B are series coils consisting of a few turns of heavy wire through which the total current flows, while C is a voltage coil of many turns of fine wire. It is connected across the load at the points H and K, and is mounted so as to turn about an axis through its geometrical center perpendicular to the plane of the paper. Attached to the axle carrying this coil, is a pair of spiral springs, not shown in the

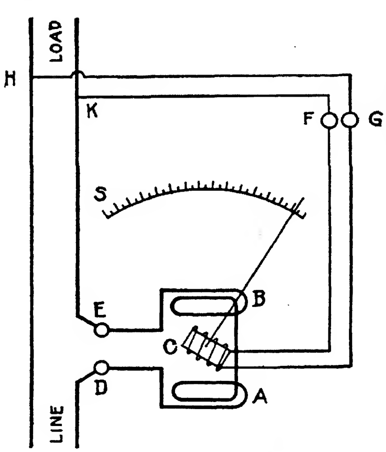

Fig. 47.—Schematic diagram for Weston wattmeter.

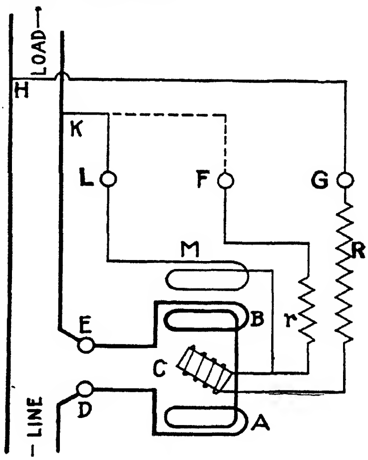

Fig. 48.—Diagram showing compensating and multiplying coils for Weston wattmeter.

figure, whose restoring torque, as the coil is rotated, opposes that due to the electrodynamic action of the currents. They serve also as leads to and from the coil. The scale is so divided that the instrument indicates watts directly.

The readings of such an instrument are subject to an error due to the power consumed by the coils themselves. An inspection of Fig. 47 shows that the current passing through the coils A and B is the sum of the load current and that carried by the coil C, hence the reading must be too large by the I^2R loss in this coil. If it is attempted to overcome this by connecting the voltage

terminals on the "line" side of the current coils, the registered voltage will be too large by the drop across the current coils thus again making the reading too large. To overcome this difficulty, a compensating coil M is provided as shown in Fig. 48, which is usually placed inside A and B and so connected that its magnetic effect weakens their fields, thus automatically correcting the reading of the instrument. If the wattmeter is to be calibrated by using separate sources of current and potential, this compensation is not necessary, and a separate binding post F is provided, marked Ind. (Independent) on the instrument. This circuit includes a resistance r equal to that of the compensating coil, thus making the resistance between C and F equal to that between C and L. The series resistance R is used as a multiplying resistance in exactly the same manner as the multiplier in an ordinary D.C. voltmeter. For example, if R is equal to the resistance of the movable coil, the potential difference across it will be equal to that across the coil, and if the instrument is calibrated without R in circuit, when R is included, the readings should be multiplied by the factor two.

65. Experiment 11. *Calibration of a Wattmeter.*—Wattmeters are calibrated on direct currents and may be used on alternating currents as well as direct. Separate sources of current and electromotive force are generally used for purposes of calibration since instruments of large capacity may then be standardized with a comparatively small expenditure of power. Connect the apparatus as shown in Fig. 49, where WM is the wattmeter which is to be calibrated. B is ten-volt storage battery furnishing the current which is controlled by the rheostat R and read by the ammeter AM. C is another storage battery furnishing the potential which is controlled by the voltage regulating rheostat PS and read by the voltmeter VM. Since the field due to the coils of the instrument is small, extraneous fields, such as those of the earth or large currents, near-by instruments with permanent magnets, etc., may cause errors as large as several per cent. Hence it is necessary, when using this type of wattmeter on direct currents, to reverse both potential and current leads and average the two readings. Make two calibrations. First, hold the current constant and vary the voltage so as to check the instrument at eight or ten points uniformly distributed across the scale. Next hold the voltage constant and vary the current, checking approximately the same points as before.

Record volts, amperes, and indicated watts, both direct and reversed, in all cases. With about two-thirds full scale deflection, bring an instrument with a permanent magnet near the wattmeter and note the effect on the reading. Place the wattmeter pointing in various directions and note any changes due to the earth's magnetic field. Stand the instrument in a vertical position and note any error due to imperfect balancing of the moving system. Change the voltage terminal from the post F, marked "Ind." to L and note the difference, which is the correction for internal energy consumption.

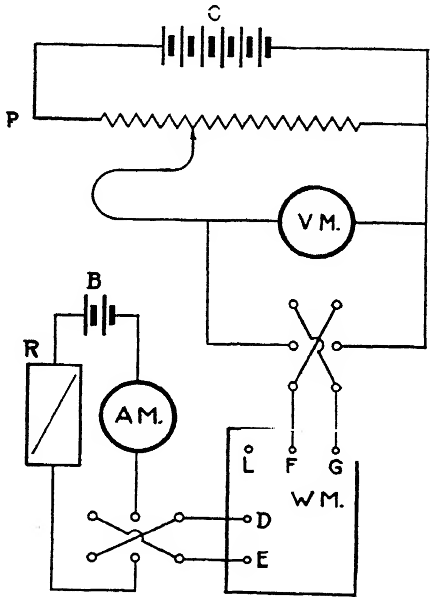

FIG. 49.—Connections for calibrating a wattmeter.

Report.—1. Compute true watts from the average products of amperes and volts.

2. Plot corrections as ordinates against wattmeter readings as abscissas. Do the two curves (a) with current constant, and (b) with voltage constant, coincide?

3. State your findings regarding internal energy consumption, effects of extraneous magnetic fields, balancing of system, etc.

4. Why are the scale divisions in this wattmeter unequal and those of the D.C. voltmeter and ammeter equal?

CHAPTER VII

MEASUREMENT OF CAPACITANCE

66. Condensers.—When a body is charged with a quantity of electricity Q, the potential V which the body acquires is proportional to Q. With a given charge, however, the potential depends also upon certain conditions of the body such as size, shape, surrounding medium, presence of other charges, etc. The relation between charge and potential is given by the equation

$$Q = CV \tag{1}$$

where C is a constant depending upon the conditions of the body, and is called the "Capacitance" of the body. It is the ratio of the charge to the potential and is numerically equal to the charge when the potential is unity. The practical unit of capacitance is the farad. A body is said to have a capacitance of one farad when a charge of one coulomb raises its potential by one volt. The farad is too large a unit for practical purposes, however, and it is customary to take the millionth part of this, called the microfarad, as a working unit. Any device by which it is possible to cause a large quantity of electricity to exist under a relatively small potential is called a condenser. Such devices usually consist of thin conducting plates, placed close together, but insulated electrically by thin sheets of some good dielectric material. If a positive charge is placed upon one plate and a negative upon the other, the neutralizing effect of each on the other, due to their close proximity, causes the potential difference between them to be very much reduced over what it would have been if they were far apart.

67. Grouping of Condensers.—Condensers, like resistances, may be joined either in series or in parallel and used as a single condenser. Figure 50 represents three condensers joined in parallel. Let C_1, C_2, C_3 represent their individual capacitances, q_1, q_2, q_3 their charges; and E, the difference of potential across their terminals. Calling Q the total quantity of electricity stored in the group, which would be obtained if they were discharged, we have

$$Q = q_1 + q_2 + q_3 \tag{2}$$

If C is the resultant capacitance of the group, we have, from definition,

$$Q = CV = C_1V + C_2V + C_3V \tag{3}$$

or

$$C = C_1 + C_2 + C_3 \tag{4}$$

For condensers connected in parallel, the resultant capacitance is the sum of the individual capacitances of the group. The capacitance of n similar condensers thus joined is n times the capacitance of a single condenser. Figure 51 represents three condensers connected in series. As before, let C_1, C_2, C_3 represent

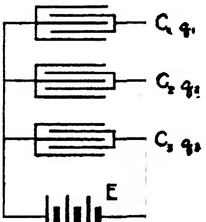

FIG. 50.—Condensers joined in parallel.

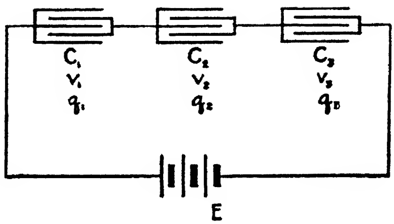

FIG. 51.—Condensers joined in series.

the values of the individual conaensers; q_1, q_2, q_3 their charges, and v_1, v_2, v_3 the potential differences across each. It is evident from the figure that

$$E = v_1 + v_2 + v_3 \tag{5}$$

Calling C the resultant capacitance of the group, and Q the total charge, we have, from definition,

$$E = \frac{Q}{C} = \frac{q_1}{C_1} + \frac{q_2}{C_2} + \frac{q_3}{C_3} \tag{6}$$

A simple relation exists between these charges. We have tacitly assumed that the condensers were uncharged before connection. Suppose a unit charge passes from the battery to the outer coating of C_1. A negative charge will then be induced on the inner coating and a positive unit charge repelled from it which will charge the outer coating of C_2 and induce a negative unit charge on its inner coating and so on. The next unit charge from the battery will do the same. It is evident then that the charge for each condenser is the same, no matter what its capacitance and that the total charge which may be obtained from the group on discharge is the same as the charge from any one condenser. That is, the

positive charge on the outer coating of C_1 neutralizes the charge on the inner coating of C_3 and similarly for the other condensers of the group. Thus we have

$$Q = q_1 = q_2 = q_3 \tag{7}$$

and

$$1 = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \tag{8}$$

For condensers joined in series, the reciprocal of the resultant capacitance is the sum of the reciprocals of the individual capacitances. The resultant capacitance of n similar condensers so joined is $\frac{1}{n}$ times the capacitance of a single condenser.

68. Standard Condensers.—Standard condensers are made of sheets of tin foil separated by mica, alternate sheets of foil being joined as shown in Fig. 50, and the whole finally imbedded in solid paraffin. Figure 52 shows the connections for

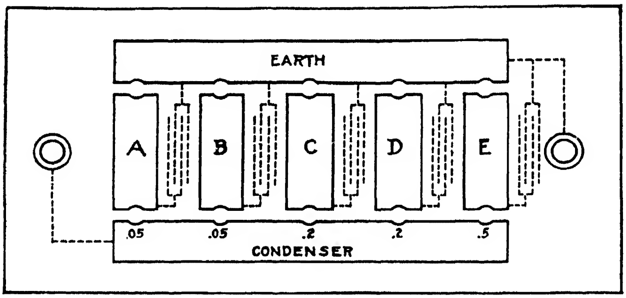

FIG. 52.—Connections for subdivided standard condenser.

one of the subdivided standard condensers used in the laboratory. One side of each of the sections shown by the dotted lines is joined to a heavy bar marked "Earth" and the other sides to one of the blocks. Another bar marked "Condenser" is mounted opposite, and each bar is connected to a binding post. When it is desired to use a certain capacity, e.g., 2 *MF*, place a plug in the socket between C and the lower bar. If .7 *MF* is desired, place another plug between E and this bar. Similarly for the various other possible connections. When a section is not in use, it should be short circuited by placing a plug in the socket between the upper bar and the corresponding block. Care should be taken never to place plugs at both ends of any block as

that would short circuit the entire condenser, possibly injuring apparatus to which it is connected.

Another method of assembling subdivided standard condensers is to join the units, not between the central lugs and one of the bus bars as shown in Fig. 52, but to connect them between the lugs as shown in Fig. 53. Thus between A and B there is .05 micro-

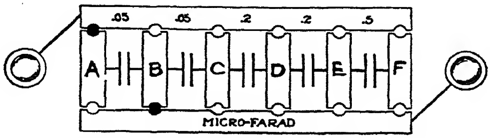

FIG. 53.—Alternate method of connecting condenser units.

farads, between B and C, .05, etc. One more lug is required in this case. To connect all the units in parallel, plugs are inserted in alternate holes on each side, but staggered. The method has the advantage of permitting series grouping of the units, thus giving a greater number of values of capacitance for a given number of units. A subdivided standard condenser is shown in Fig. 54.

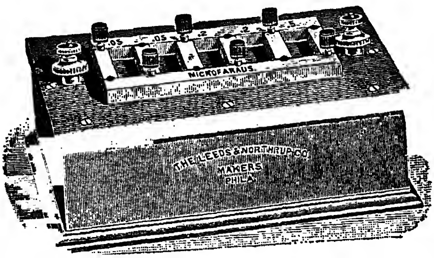

FIG. 54.—Subdivided standard condenser.

69. Comparison of Condensers.—The capacitance of an unknown condenser may be found by comparing it with a standard condenser. A ready means of doing this which gives results sufficiently accurate for many purposes is to charge both the unknown and the standard to the same potential difference and discharge each in succession through a ballistic galvanometer. The set-up for this purpose is shown in Fig. 9. Let C_1 be the unknown and C_2 the standard condenser. First insert C_1, then charge and discharge several times, taking the average deflection

which we will call d_1. From the definition of capacitance we have, as the charge in the condenser.

$$Q_1 = C_1V, \tag{9}$$

and, since the deflection of a ballistic galvanometer is proportional to the charge passed through it,

$$Q_1 = C_1V = Kd_1 \tag{10}$$

Now replace the unknown by the standard condenser and charge and discharge as before. In a similar manner, we have

$$Q_2 = C_2V = Kd_2 \tag{11}$$

Dividing equation (10) by (11),

$$\frac{C_1}{C_2} = \frac{d_1}{d_2}. \tag{12}$$

70. Bridge Method for Comparing Two Condensers.[1]—A more accurate comparison of two condensers may be made by means

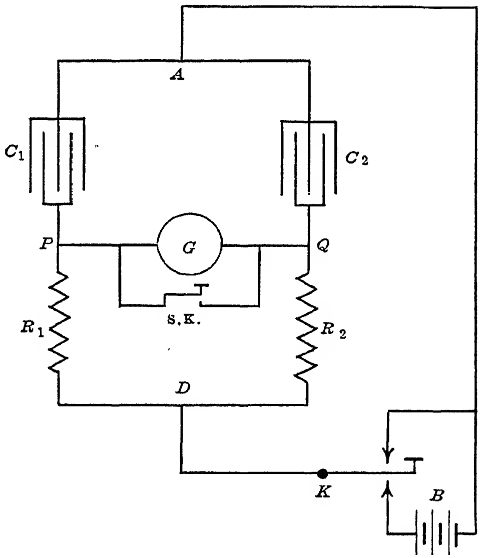

FIG. 55.—Bridge method for comparing two condensers.

of an arrangement similar to the Wheatstone bridge in which the resistances of two of the arms are replaced by the two condensers to be compared, and the current galvanometer is replaced by a

[1] CARHART and PATTERSON, Electrical Measurements, pp. 213–220. SMITH, Electrical Measurements, art. IX.

ballistic galvanometer. The connections are shown in Fig. 55. C_1 and C_2 are two condensers and R_1 and R_2 two variable resistance boxes. K is a charge and discharge key so connected that, when the blade is pressed down, the battery B is connected to the bridge, thus charging the condensers through R_1 and R_2 to the potential difference furnished by the battery. When contact is made on the upper point, the battery is disconnected and the condensers are discharged through the resistances. No matter what the values of R_1 and R_2 may be, the points P and Q will come to the same final potentials on charge and again on complete discharge, since, when no current is flowing through the resistances there can be no fall of potential across them. However, during the charging and discharging processes there are currents through the resistances and, in general, there will be a momentary difference of potential between P and Q causing a deflection of the galvanometer in one direction on charge, and in the opposite, on discharge. By properly adjusting R_1 and R_2, it is possible to make the potentials at P and Q rise and fall at the same rate which is the balance condition for the bridge, from which the relation between the capacitances and resistances may be deduced.

Let q_1 and q_2 = instantaneous charges in C_1 and C_2
Let i_1 and i_2 = instantaneous currents in R_1 and R_2

As in the ordinary Wheatstone bridge, we have

P.D. between A and P = P.D. between A and Q
P.D. between P and D = P.D. between Q and D

whence

$$\frac{q_1}{C_1} = \frac{q_2}{C_2} \tag{13}$$

and

$$R_1 i_1 = R_2 i_2 \tag{14}$$

Differentiating (13) and remembering that $i = \dfrac{dq}{dt}$, we have

$$\frac{1dq_1}{C_1dt} = \frac{1dq_2}{C_2dt}$$

or

$$\frac{i_1}{C_1} = \frac{i_2}{C_2} \tag{15}$$

Dividing (14) by (15)

$$R_1C_1 = R_2C_2 \tag{16}$$

or

$$\frac{C_1}{C_2} = \frac{R_2}{R_1} \tag{17}$$

It is to be noted that the ratio of the capacitances is the inverse ratio of the resistances, whereas, in the bridge method for resistances, it is the direct ratio.

71. Experiment 12. *Comparison of Two Capacitances by the Bridge Method.*—Connect the apparatus as shown in Fig. 55, using for B a battery of 40 volts. SK is a short circuiting key to bring the galvanometer to rest after taking an observation. Use for C_1 a subdivided standard condenser and for C_2 a fixed condenser of about one-half micro-farad capacity. The problem is to check the parts of C_1 in terms of the whole. Set the plugs of C_1 so as to give the maximum available capacitance, and with this as a standard, obtain several balances on C_2, using different values for R_1 and R_2 in each case. Now, taking this measured value of C_2 as a standard, determine the capacitance of each part of C_1, making several independent balances for each. In all cases use as large values for R_1 and R_2 as possible.

Report.—Tabulate your data in compact form. Your values for the separate parts of C_1 should add up to the total value indicated on the top of the box.

72. Measurement of Small Capacitance by Commutator.[1]—The bridge method just described is not suited to the measurement of small capacitances since the charging currents are so minute that the potential drops through the resistances are inappreciable. For the determination of a small capacitance, such as that of an air condenser used in radio work or that between the wires of a transmission line, a direct deflection method may be used in which the condenser is rapidly charged to a known voltage and then discharged through a standardized galvanometer by means of a motor-driven commutator. If the interval between discharges is small compared to the period of the galvanometer, a steady deflection results which is proportional to the average value of the current.

[1] FLEMING and CLINTON, *Proc. Phys. Soc. of London*, vol. 18, 1901–03, p. 386.

Figure 56 shows the principle of the Fleming and Clinton commutator, designed for this purpose, together with the wiring diagram. S_1 and S_2 are slip rings which revolve in a plane perpendicular to the paper while S_3 is a series of posts alternately connected to S_1 and S_2. When brush 3 rests upon a segment connected to S_1, the condenser C is charged to the potential difference of the battery B, and when 3 touches the succeeding post, the condenser is discharged through the galvanometer. Let n be the number of discharges per second and V the E.M.F. of the battery. Then the current through the galvanometer is

$$I = nCV \times 10^{-6} = Fd \qquad (18)$$

Where C is the capacity of the condenser in microfarads, and F, the figure of merit of the galvanometer, i.e., the current for unit deflection.

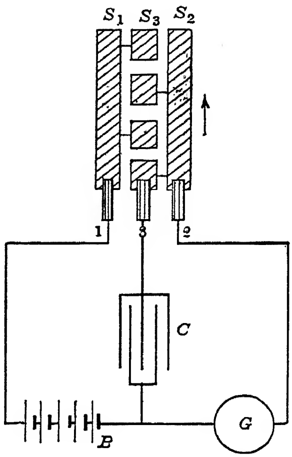

FIG. 56.—Fleming and Clinton commutator.

In designing a commutator of this type, special precautions must be taken to secure good insulation between posts and sectors. They are generally made with an air gap between posts since metal abraded by friction otherwise embeds itself in a solid dielectric thus giving a direct leakage path from the battery through the galvanometer. A speed counter, mounted on the shaft, indicates the number of revolutions.

73. Experiment 13. *Capacitance by the Fleming and Clinton Method.*—Connect the apparatus as shown in Fig. 56 using for C an air condenser of small capacitance. Drive the commutator at speeds sufficient to give 50 to 100 discharges per second through the galvanometer, and use for B a battery with voltage large enough to produce a deflection of about 10 cms. Take special precautions to insure good insulation between the galvanometer and battery circuits. Use a number of different speeds and voltages and determine the value of C by eq. (18). Determine the figure of merit of the galvanometer by the method given in Art. **23**.

Report.—1. Tabulate observations and results for the series of measurements taken.

2. Check your results by measuring the dimensions of the condenser, and computing its capacity from the formula for the parallel plate condenser

$$C = .0885 \times 10^{-6} \frac{KA}{d} \text{ microfarads} \tag{19}$$

where A is the area of one of the plates in square centimeters, d the thickness of the dielectric, and k the dielectric constant ($K = 1$ for air).

3. How do you account for the difference between the measured and computed values?

CHAPTER VIII

MAGNETISM

74. General Principles.—Magnetism is a universal property of matter, since there is no substance which does not experience a ponder-motive action when placed near the poles of a strong magnet, though in many cases the effect is so weak that delicate means are necessary for its detection. Substances may be divided into two groups, in accordance with the manner in which they behave when acted upon by a magnetic pole; those which are attracted by the magnet are called paramagnetic, and those repelled, diamagnetic. It is customary to add a third group, including iron, nickel, and cobalt, which are characterized by the fact that the ponder-motive action upon them is not proportional to the strength of the attracting pole as in the case of ordinary paramagnetic substances, but is much stronger.

75. Strength of Pole.—In the early study of magnets, it was noticed that the magnetic property of a body is confined largely to the areas about its ends and corners, and that opposite ends behave differently toward other magnets. The term magnetic pole was given to the regions where the property was most pronounced and has been retained although it has been known for a long time that magnetism is a volume and not a surface phenomenon. *A unit magnetic pole is defined as one which repels an exactly similar pole at a distance of one centimeter in air with a force of one dyne.*

76. Strength of Field.—The space surrounding a magnetic pole in which action upon another pole can be detected is called a magnetic field, and is measured, at any particular point, by the force in dynes with which a free unit positive pole is acted upon when placed at that point. A *field of unit strength or intensity is one which will exert a force of one dyne upon a unit pole.* Since field strength is thus measured by force per unit pole, it is a vector quantity; *i.e.*, it possesses both magnitude and direction. Both of these characteristics may be represented by imagining lines drawn in space according to a definite convention; namely,

the magnitude of the field by drawing as many lines per square centimeter as the field has units of intensity, and the direction, by making these lines coincide at every point with the direction in which the unit measuring pole is urged. According to this convention, if a sphere of unit radius is drawn with a pole of strength m as a center, there must pass through each square centimeter of its surface m lines, giving a total of $4\pi m$ lines from the pole. From a unit pole there will emanate according to this convention, 4π lines of force.

77. Intensity of Magnetization.—Let us imagine an ideal permanent bar magnet, of length L, and uniform cross section A, magnetized uniformly and showing, therefore, pole-strength over the ends only. That is to say, the magnetic lines all leave one end, pass in regular curves through the outside space, and enter the other end with no lines entering or leaving on the side, as in any real magnet. Let the strength of pole be m. The pole strength per unit area, $\frac{m}{A}$, is defined as the intensity of magnetization and is generally represented by the letter I.

78. Magnetic Moment.—Imagine the ideal magnet mentioned above placed at right angles to a uniform magnetic field of strength H. Equal and opposite forces of magnitude Hm will act upon this magnet producing a couple of strength HmL. If H is unity, the magnitude of the couple is mL, and this quantity, which is exceedingly important in treating problems involving magnets, is called the magnetic moment, and is designated by the letter M. The moment of any magnet is, then, the torque acting upon it when placed at right angles to a uniform field of unit strength. Another definition of intensity of magnetization in terms of magnetic moment may be obtained as follows: Since the volume of the bar magnet is LA, we have

$$I = \frac{m}{A} = \frac{mL}{AL} = \frac{M}{V} \tag{1}$$

Intensity of magnetization is thus defined as a volume rather than a surface effect.

79. Magnetic Induction.—Let us imagine that, in an infinitely long, uniform magnetic field of strength H, an iron bar is placed with its axis parallel to the field. The bar becomes magnetized to an intensity I and is equivalent to the ideal magnet considered above. The number of magnetic lines through the space

occupied by the bar has been increased by the lines of magnetization due to the bar. The total number of magnetic lines through the bar, which is made up of the original lines and the lines of magnetization, is called the magnetic flux, and is generally designated by the Greek letter ϕ. The number of lines per square centimeter through the bar is called the magnetic induction, and is represented by the letter B. Thus

$$B = \text{Induction} = \frac{\text{Total Flux}}{\text{Area}} = \frac{\phi}{A} \tag{2}$$

The induction B is defined in the following manner: Imagine a narrow crevasse cut through the middle of the bar at right angles to its axis, and a unit positive pole placed within. The force in dynes upon this pole measures B. The original field produces a force of H dynes upon the pole, and since the iron is magnetized to an intensity I, meaning I units of pole strength per unit area of the crevasse, from each of which 4π lines emanate, we have, as the total lines per square centimeter through the gap or the force in dynes acting upon the unit pole

$$B = H + 4\pi I \tag{3}$$

Lines of induction are continuous throughout the magnetic circuit; that is, they never begin or end but form closed paths, the parts in the air being called lines of force. If, instead of the transverse crevasse we had bored a small hole through the bar parallel to the lines of force and placed a unit magnetic pole within, the force upon it would be the original strength of field H which has produced the magnetic induction.

80. Permeability and Susceptibility.—For many purposes it is convenient to define the magnetic quality of a given material in terms of the relative increase in the number of lines or the intensity of magnetization produced. For this purpose the terms *permeability* and *susceptibility* are used. By permeability is meant the ratio of the induction B to the field strength H, and is represented by the Greek letter μ. That is,

$$\mu = \frac{B}{H} \tag{4}$$

where B is the induction produced in a given material when acted upon by a field of strength H. When it is desired to express the ability of a material to acquire magnetism and to state its condition in terms, not of the total induction, but of its own magnetic lines alone, we use the term *susceptibility*. This is

defined as the ratio of the intensity of magnetization of the specimen to the magnetizing field in which it is placed, and is represented by the Greek letter κ. That is,

$$\kappa = \frac{I}{H} \tag{5}$$

A simple relation exists between these two quantities. Taking the defining equation for induction

$$\mathrm{B} = \mathrm{H} + 4\pi\mathrm{I} \tag{6}$$

and dividing through by H, we have

$$\frac{B}{H} = 1 + 4\pi\frac{I}{H} \tag{7}$$

or

$$\mu = 1 + 4\pi\kappa \tag{8}$$

81. Effects of the Ends of a Bar.—When a bar is magnetized longitudinally by placing it in a magnetic field, the ends become poles which act upon any other pole in the neighborhood, attracting or repelling it according to the relative signs of the poles. If the bar lies in an east-west position, magnetized with a north pole at the west end, a unit north pole lying near the middle of the bar, but outside it, would be urged from west to east or in a direction opposite to that in which the bar is magnetized. If now the unit pole is placed within the bar, the force is in the same direction. Thus the effect of the poles is to produce a field within the bar called a "demagnetizing field" which is opposite to the direction of the field magnetizing it. This effect is greater the shorter the bar is in comparison to its diameter. The actual field producing magnetization is, accordingly, less than the field before the bar was introduced. This phenomenon is allowed for by computing the effective field H from the equation

$$H = H' - NI \tag{9}$$

where H' is the original field and N a constant depending upon the ratio of the length to the diameter of the bar, and is called the "Demagnetizing Factor." Tables[1] for N may be found in the more advanced treatises on the subject. The same considerations hold for solenoids, and hence it is necessary, when one wishes a solenoid whose field may be computed readily from its dimensions, to make it long in comparison to its diameter. If one uses a ring solenoid, or a test specimen in the form of a ring,

[1] Du Bois, The Magnetic Circuit, p. 41.

this correction is unnecessary since there are no free poles to produce disturbing effects of this character.

82. The Magnetic Circuit.—In treating such phenomena as the conduction of heat and the flow of electricity, one makes use of a general law in which the magnitude of the effect is given as the ratio of a driving force divided by an opposition factor dependent upon the properties of the medium in which the action takes place. For example, the heat current Q, *i.e.*, the quantity of heat passing per unit time any cross section of a conductor of length L and cross sectional area A, when the temperature at the ends are t_1 and t_2, is given by the expression

$$Q = \frac{t_1 - t_2}{\frac{L}{\tau A}} \tag{10}$$

where τ is a constant defining the ability of the medium to conduct heat. τ is called the specific thermal conductivity and is numerically equal to the quantity of heat passing through a centimeter cube of the material, per unit time, when a difference of temperature of one degree is maintained across its faces. Similarly, the electrical current flowing in the above conductor when its ends are maintained at electrical potentials V_1 and V_2, is given by

$$I = \frac{V_1 - V_2}{\frac{L\rho}{A}} = \frac{V_1 - V_2}{\frac{L}{CA}} \tag{11}$$

where $C = \frac{1}{\rho}$ is called the specific electrical conductivity of the material and is numerically equal to the current flowing through a centimeter cube when unit difference of potential is maintained across its faces. Its reciprocal ρ is called the specific resistance, and is the resistance of the centimeter cube. This equation is called Ohm's law and is written

$$\text{Current} = \frac{\text{Electromotive Force}}{\text{Resistance}}$$

In an analogous manner it is convenient, for purposes of calculation, to regard the region in which a magnetized state exists as being the seat of a magnetic flow. The magnetic lines of induction are the stream lines along which the flow takes place, and since magnetic lines are closed paths, the lines of flow are closed circuits. Materials are then classified as good or bad magnetic conductors according to the ease with which they are magnetized.

To make the analogy clear, consider a specimen of magnetic material in the form of an anchor ring, wound uniformly with wire through which a current is flowing, as shown in Fig. 57. We wish to compute the total magnetic flux produced in the ring when a given current is flowing.

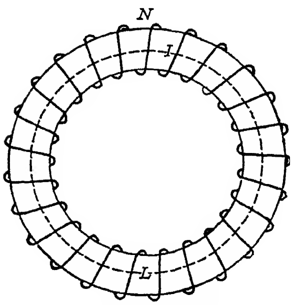

FIG. 57.--The magnetic circuit.

Let N = total number of turns
L = mean length of magnetic lines
A = area of cross section of ring
B = magnetic induction in ring
μ = permeability
I = strength of current

As a direct consequence of the definition of the electromagnetic unit of current, it is shown in elementary textbooks that the work done in carrying a unit magnetic pole once around a current of strength I E.M.U.'s, is 4πI ergs. The field strength within the ring solenoid may be obtained from the fact that the work done in taking a unit pole around this magnetic circuit is

$$\text{Work} = HL = 4\pi NI \tag{12}$$

since the pole is, in reality, carried N times around the current. Whence

$$H = 4\pi NI \tag{13}$$

If I is expressed in amperes instead of electromagnetic units,

$$H = \frac{4\pi NI}{10L} \tag{14}$$

From the above definitions, the expression for the total flux is obtained in the following manner:

$$\phi = BA = \mu HA = \frac{\mu A 4\pi NI}{10L} \tag{15}$$

which may be written in the form

$$\phi \quad \frac{.4\pi NI}{\frac{L}{\mu A}} \tag{16}$$

The numerator of the right-hand member is of the nature of a driving force, the denominator, an opposition factor depending upon the medium, and their ratio, the effect produced. This

equation is called the "Law of the Magnetic Circuit" and is written

$$\text{Magnetic Flux} = \frac{\text{Magnetomotive Force}}{\text{Reluctance}}$$

83. Magnetic flux, which represents the total number of lines of induction, is analogous to flow of heat in calorimetry, and to current in electricity. It forms a closed path which may be spread out over a large area in some places and be concentrated within narrow limits in others. The unit of magnetic flux is called the maxwell and is represented by one magnetic line of induction through the total cross sectional area of the magnetic circuit. Thus, if in a magnetic circuit, there are one thousand lines, the flux is said to be one thousand maxwells. In engineering practice, it is customary to define flux on the basis of the E.M.F. induced in a conductor which cuts it.

Definition.—If, in a moving conductor, the induced E.M.F. is one electromagnetic unit, the flux cut per second is one maxwell.[1]

84. The magnetic induction is defined as the total flux divided by the area, and is, accordingly, the flux density. The unit of magnetic induction is the gauss.

Definition.—Unit induction, or one gauss, exists in a magnetic circuit in which the flux density is one maxwell per square centimeter. Thus

$$\text{Gausses} = \frac{\text{Maxwells}}{\text{Square Centimeters}}$$

85. Magnetomotive force may be regared as the cause of magnetic flux. It is analogous to electromotive force in the electric circuit and is measured in a similar manner. Just as the electromotive force of an electrical circuit is the work required to carry unit electrical charge once around the circuit, so the magneto-motive force in a magnetic circuit is the work required to carry unit magnetic pole once around the circuit. The unit of magnetomotive force is the gilbert.

Definition.—If the work required to carry a unit magnetic pole once around a magnetic circuit is one erg, the magnetomotive force is one gilbert.

In case the magnetomotive force is produced by a current in a closed solenoid, as in the above illustration, its value, as given

[1] Note.—The Units for the quantities involved in the magnetic circuit here described were adopted by the International Electrical Congress at Paris, in 1900.

by equation (16) is $.4\pi$NI. The product NI is called the ampere turn, and differs from magnetomotive force only by the constant factor $.4\pi = 1.26$. Thus

$$\text{M.M.F. in Gilberts} = .4\pi \text{ Ampere Turns.}$$

Magnetomotive force, being thus measured in terms of work per unit pole, is difference of magnetic potential. Accordingly, if H is the average value of the magnetic field strength between two equipotential surfaces, s cms. apart, having magnetic potentials M_1 and M_2, respectively,

$$H = \frac{M_1 - M_2}{s} = \frac{\Delta M}{\Delta s} \tag{17}$$

where ΔM and Δs represent small differences in M and s, respectively. Allowing the equipotential surfaces to approach indefinitely close to one another, the limiting value of this ratio gives the actual field strength at a given point. Thus

$$H = \frac{dM}{ds}. \tag{18}$$

Magnetic field strength is the change in magnetic potential per centimeter in the direction of H or the magnetic potential gradient. The unit of magnetic field strength is called the gilbert per centimeter.

86. Reluctance is the resistance a body offers to being magnetized and depends upon the constants of the circuit in a manner similar to resistance in the electrical circuit. As seen from eq. (16), it is directly proportional to the length and inversely proportional to the area and the permeability of the medium. Permeability thus corresponds to specific conductivity, and its reciprocal, corresponding to specific resistance or resistivity, is often called "reluctivity." The unit of reluctance is defined in terms of the law of the magnetic circuit and is called the oersted.

Definition.—If, in a magnetic circuit, the flux is one maxwell when the magnetomotive force is one gilbert, the reluctance is one oersted.

Reluctances, like resistances, may be joined in series or parallel to form complex circuits, and laws similar to those for resistances hold.

1. For reluctances joined in series, the total reluctance is the sum of the individual reluctances.

2. For reluctances joined in parallel, the reciprocal of the total reluctance is the sum of the reciprocals of the individual reluctances.

87. Limitations.—While the idea of the magnetic circuit is an extremely useful one for purposes of calculation, it must not be regarded as a true physical concept, such as the electrical circuit, but merely as an analogy serving a useful purpose. Among the respects in which the analogy fails are the following:

1. There is no such thing as a magnetic substance in the sense in which we have used it, and hence there can be no magnetic flow.

2. When once the magnetic flux has been established, no energy is required to maintain it, and there is nothing corresponding to the I^2R consumption of energy in the electric circuit.

3. The reluctance of a circuit containing ferro-magnetic material is not a constant for a given set of physical conditions but varies with the flux, while the resistance of an electric circuit is independent of the current flowing.

4. For ferro-magnetic materials, the reluctance is not a single valued function of the flux, but depends upon the magneto-motive forces to which they previously have been subjected. In other words, there is no analogy, in the electric circuit, to Hysteresis.

88. Magnetization Curves.—Para-and diamagnetic substances are characterized by the fact that, under a given set of physical conditions, the permeability remains constant; that is, as the magnetizing field is changed, the induction changes by proportional amounts. This, however, is not true of ferro-magnetic substances. If a piece of unmagnetized iron, for example, is placed in a field which may be varied at will, it is found, starting with $H = O$ and gradually increasing it, that the induction B increases slowly at first, remaining nearly proportional to the field; then increases rapidly, for a certain interval of H, after which a further increase produces only relatively small increments in B. The curve showing the values of induction for different magnetizing fields is called the "magnetization curve," and is represented by OB of Fig. 58. The three parts of the curve, differentiated by rather abrupt changes in slope, are accounted for by assuming that, in the unmagnetized condition, the magnetic axes of the molecular magnets are distributed entirely at random, as many pointing in one direction as in any other; and that the magnetic circuits, of which they form parts,

are small closed curves. Under the action of a weak magnetic field, these molecular magnets are all sprung to a slight extent from their initial positions, giving a resultant component in the direction of the applied field, the amount of deformation being proportional to the field. Thus the part of the curve near the origin is obtained. With a further increase of field, some of these local magnetic circuits are broken, and new alignments formed, giving chains of molecules of considerable length. As each local circuit breaks, becoming part of a chain, neighboring

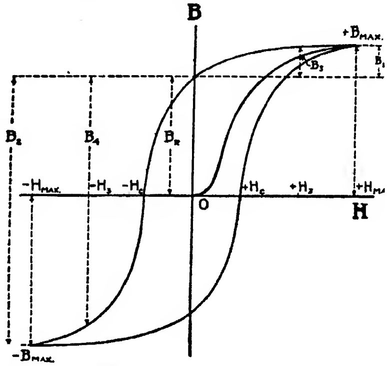

FIG. 58.—Magnetization and hysteresis curves.

groups become unstable, break, and form other chains, thus giving a sort of spontaneous magnetization, resulting in changes in induction much greater than required for proportionality to changes in field. Thus the steep part of the curve is given. As the condition is approached in which all the local groups have been broken up and the molecules placed in complete alignment, the iron is said to become saturated, and further increases in field produce only small changes in induction. So the upper part of the curve which is nearly horizontal is obtained.

89. Hysteresis.—If, after the induction has been carried to the point marked $+B_{\text{max}}$ on the curve of Fig. 58, the magnetizing field is gradually reduced, the induction does not retrace the magnetization curve, but takes on values, for a given field, greater than those for the magnetization curve; and when H has been reduced to zero, an amount of induction indicated by B_R still persists. If a reverse field is applied, the induction rapidly falls; and when a certain value, $-H_c$, has been reached, the resultant induction is zero; after this, a further negative increase in field to

$-H_{max}$ gives a reversed value of induction $-B_{max}$ equal in magnitude to $+B_{max}$. With a gradual increase in H to its original positive value, B assumes values shown by the lower curve of the figure, symmetrical with respect to the origin with the upper one just described. This tendency of any material to persist in a given magnetic state is known as "hysteresis," and the corresponding curve is called the hysteresis curve. B_R is called the retentivity, and H_C, the coercive field.

It may be shown that the area of the hysteresis loop is a measure of the energy consumed by molecular friction in each cubic centimeter of material when carried once through a magnetic cycle. For this purpose, let us refer to the ring specimen described in Art. 82, and use the nomenclature there indicated. The method of proof is based upon the fact that, as the current in the magnetizing coil is changed, producing changes of flux in the ring, a counter E.M.F. is induced, against which the magnetizing current must flow. The electrical energy which thus disappears is the energy consumed by hysteresis and reappears in the form of heat within the ring. Let i represent the instantaneous magnetizing current and let dB and $d\phi$ be the changes in induction and flux, respectively, when a change di occurs in the magnetizing current. If dt represents the time required for this change to take place, the energy dw consumed during the change is given by

$$dw = eidt \tag{19}$$

But

$$e = N\frac{d\phi}{dt} = NA\frac{dB}{dt} \tag{20}$$

Therefore

$$dw = NAidB \tag{21}$$

Since

$$H = \frac{4\pi Ni}{L},$$

we have

$$Ni = \frac{HL}{4\pi}. \tag{22}$$

Substituting

$$dw = \frac{HLAdB}{4\pi} = \frac{V}{4\pi}HdB \tag{23}$$

where V is the volume of the ring. Summing up for the complete cycle, we have

$$\int dw = W = \frac{V}{4\pi}\int_c HdB = \frac{V}{4\pi}\text{ (area of loop)} \tag{24}$$

It is thus seen that the area of the loop divided by 4π gives the energy lost per cycle per cubic centimeter of material. The shape of the loop varies with the quality of the iron; hard steels have both a high retentivity and coercive force; soft steels, a high retentivity but a low coercive force; while Swedish iron has both a low retentivity and low coercive force. For a given specimen, the area of the loop depends upon the limits of induction. Steinmetz has made an exhaustive study of this relation and has found that the energy lost is proportional to the 1.6 power of the maximum induction. Expressed in symbols,

$$\frac{W}{V} = KB_m^{1.6} \tag{25}$$

K is called the Steinmetz coefficient.

90. Practical Methods.[1]—For the measurement of magnetic induction, there are three general methods, each of which possess certain advantages as well as disadvantages. They may be classified as follows:

1. The Traction Method.
2. The Magnetometer Method.
3. The Ballistic Method.

The first method consists in measuring the mechanical force required to pull the magnetized specimen away from a massive piece of iron. Since the specimen induces in the block at the point of contact a pole of strength equal and opposite to its own, the force required to separate them is proportional to the square of the intensity of magnetization. In the second method, the specimen is made in the form of a rod or elongated ellipsoid and magnetized by being placed within a long solenoid. Its magnetic moment is determined by observing the deflection it produces upon a small compass needle, called a magnetometer, placed near it. From the magnetic moment, the intensity of magnetization, and hence the induction, may be computed. In the ballistic method, the specimen under test forms the whole or part of a closed magnetic circuit, wound with suitable magnetizing coils, and also a secondary coil, connected to a ballistic galvanometer. Any change in flux induces in the secondary a quantity of electricity which is measured by the ballistic galvanometer and from this quantity the change in flux is computed. From the standpoint of accuracy and ease of performance the

[1] Ewing, Magnetic Induction in Iron, chap. II.
DuBois, The Magnetic Circuit, chap. XI.

ballistic method is much to be preferred and is the only one which will be considered here.

91. Hopkinson's Bar and Yoke.[1]—This is an application of the ballistic method in which the samples to be tested are in the form of rods, closely fitted into holes in a heavy yoke of soft iron. The arrangement is shown in Fig. 59, where YY is the yoke and CC' the specimen under test. MM are the magnetizing coils and F the secondary coil. The magnetic lines through the specimen return, half through the upper and half through the lower part of the yoke. Since the cross section of the yoke is large in comparison with that of the specimen, its reluctance

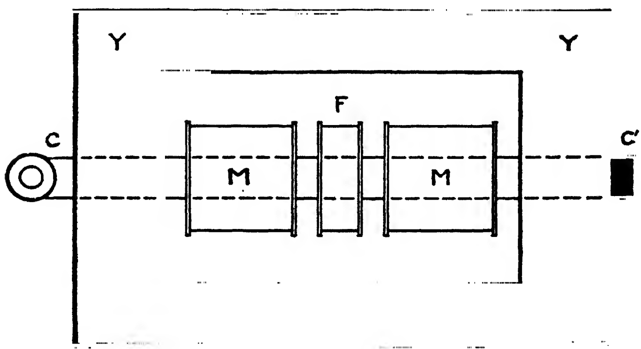

FIG. 59.—Hopkinson's bar and yoke.

may be neglected without appreciable error and the entire reluctance be considered as that part of the bar within the slot. The rod consists of two parts joined at a point a little to the right of F, one of which is clamped at C', while the other may be drawn out by the ring at C. Springs are attached to the secondary coil F, generally called the "flip" coil, which runs between guides. While the bar is being subjected to the desired magnetizing field, the part C is quickly withdrawn, releasing F which is jerked suddenly out of the field, cutting the entire flux through the specimen. The induction B, for a given value of H, is computed in the following manner:

[1] Ewing, Magnetic Induction in Iron, p. 67–92.
Smith, Electrical Measurements, chap. X.

Let K = constant of the ballistic galvanometer
n_f = turns on flip coil
R = total resistance of secondary circuit
i = instantaneous current in secondary circuit
e = instantaneous E.M.F. induced in secondary circuit
d_f = deflection of galvanometer
ϕ = total flux in specimen
A_s = area of specimen
N_m = magnetizing turns
I_m = magnetizing current
L_R = length of rod (inside length of yoke)

The quantity Q of electricity, expressed in coulombs, discharged through the galvanometer is given by the expression

$$Q = Kd_f = \int idt \tag{26}$$

But

$$i = \frac{e}{R} = \frac{n_f}{10^8R}\frac{d\phi}{dt} = \frac{n_fA_S}{10^8R}\frac{dB}{dt} \tag{27}$$

Hence

$$Kd_f = \frac{n_fA_S}{10^8R}\int_B^0 dB = \frac{n_fA_SB}{10^8R} \tag{28}$$

Therefore

$$B = \frac{KR10^8}{n_fA_S}d_f. \tag{29}$$

The constant K of the ballistic galvanometer may be obtained by means of the standard solenoid described in Art. **27.** Substituting the value of K from Eq. 36.

$$B = \frac{8\pi NnA}{10Ln_fA_S}\cdot\frac{I}{d}\cdot d_f. \tag{30}$$

The field strength to which the specimen is subjected is given by the regular formula for the ring solenoid

$$H = \frac{4\pi N_mI_m}{10L_R} \tag{31}$$

92. Experiment 14. *Magnetization Curves by Hopkinson's Bar and Yoke.*—Connect the apparatus as shown in Fig. 60, where C and Y are the bar and yoke, respectively. G is a ballistic galvanometer and DE a standard solenoid for calibrating it. Since a considerable range of currents will be required, use two ammeters, one of range 0–5 amperes and the other, a millammeter connected as shown, where S_1 is a knife switch which should be left closed during all manipulations. Open S_1 when it is

desired to read the millammeter and then only when the 0–15 ammeter indicates a current less than the full scale reading of the millammeter. S_2 is also a knife switch. First compute, by means of eq. (31), the upper and lower limits of current required for field strengths ranging from 1 to 120 gilberts per centimeter.

Before proceeding to test a specimen, it must first be

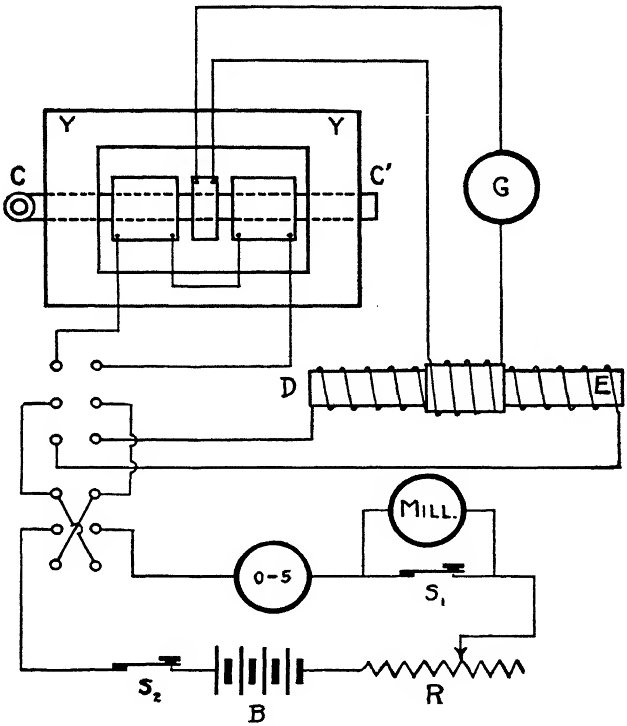

FIG. 60.—Connections for Hopkinson's bar and yoke.

demagnetized. This is done by applying a magnetizing current, somewhat greater than that required for the maximum test field, and reducing it by small steps, reversing the commutator at each step until a current barely readable on the millammeter has been reached. The rate of commutation should not exceed 20 reversals per minute. In this way, the specimen is magnetized first in one direction, and then in the other, each time to a less

extent, until finally all magnetism has disappeared. This should be done with the flip coil out of position or with the secondary circuit broken, to avoid damaging the galvanometer. To test for residual magnetism, flip the coil with no current in the magnetizing coils. A deflection not exceeding a millimeter should be obtained. Next, determine the constant of the galvanometer. To do this, set the double pole double throw switch so as to connect in circuit the primary of the standard solenoid, and, with a steady current of about two amperes flowing, reverse the commutator in such a direction as to cause the galvanometer to swing to high numbers. Make several determinations in this manner, using such values of primary current as will give deflections ranging from 2 to 14 centimeters. It is necessary here to reverse the primary current, not simply to make or break it, since that is the assumption on which the formula for the determination of the constant was derived. Use the average value of the ratio of current to deflection in eq. (30).

Everything is now ready for the test proper. Set the double pole double throw switch again so as to include the magnetizing coils and the rheostat so as to include the maximum resistance. Close the battery circuit and bring the current up to the smallest value computed above. Flip the coil and note the deflection of the galvanometer, which should swing in the same direction as used when determining its constant. Obtain, in this manner, about fifteen points on the magnetization curve, spaced more closely together in the lower part of the field strength range, where the curve rises steeply. Caution.—Points must be taken always with increasing field strength. Do not allow the current to rise too high and then decrease it. Obtain data for the magnetization curves for the samples of iron furnished. Check your galvanometer constant before and after taking each set.

Report.—1. Plot magnetization curves for the four samples using B as ordinates and H as abscissas.

2. Calculate the permeability for each value of H and, on a separate sheet, plot permeability as ordinates and field strength as abscissas for each sample.

3. For the maximum field strength, compute the magnetomotive force, total flux and reluctance of the magnetic circuit for each sample, expressing each quantity in its proper units. What is the relation between maxwells and gausses?

93. The Rowland Ring.[1]—In the bar and yoke method described above, errors are introduced due to imperfect magnetic contact between the ends of the rods and the rod and yoke. This objection is overcome by making the specimen in the form of a ring, either turned true in the lathe from a solid block, or built up of sheet stampings. The magnetizing coil is then wound uni-

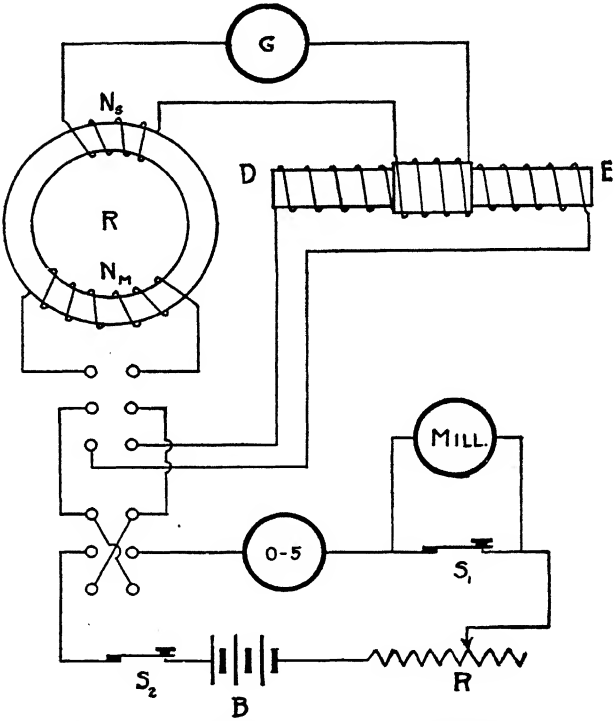

FIG. 61.—Connections for Rowland Ring Method.

formly over the entire magnetic circuit with the secondary wound over the primary. Since the turns on the inner side of the ring are closer together than on the outer, the former part of the ring will be subject to a greater magnetizing force than the latter, and therefore, the thickness of the ring should be small compared to its

[1] EWING, Magnetic Induction in Iron, chap. III.
SMITH, Electrical Measurements, chap. XII.
ROWLAND, *Phil. Mag.* vol. 46, 1873, p. 151.

diameter, the requisite area being obtained by increasing the height. As the magnetic circuit cannot be broken, it is impossible to obtain any measurement of the magnetic state of a given ring, so the method of observation is limited to measurements of changes of magnetic state produced by definite changes in magnetizing force.

A magnetization curve may be obtained by a series of reversals carried out in the following manner: Suppose the iron to be in an unmagnetized condition. Apply a weak magnetizing field. The induction rises a small amount along the desired magnetization curve. Reverse the magnetizing field. This causes the induction to change along the upper half of a small hysteresis cycle, taking on a value the negative of what it had before the reversal occurred. The change of induction, which is measured by the ballistic galvanometer, is then twice the total induction existing before the reversal. Now increase the field to a somewhat larger value, thus carrying the induction to a higher point on the curve. By again measuring the change of induction on reversal, twice the new value of induction is obtained, and so on for a series of points determining the entire magnetization curve. By this process, the iron is taken around a series of successively larger and larger hysteresis cycles, the apexes of whose corresponding curves lie upon the desired magnetization curve. The induction B, for a given value of H, is computed in the following manner:

Let K = constant of the ballistic galvanometer
N_S = turns on secondary coil
R = total resistance of secondary circuit
e = instantaneous E.M.F. in secondary circuit
i = instantaneous current in secondary circuit
d_S = deflection of galvanometer
ϕ = total flux in ring
A_R = cross sectional area of ring
N_m = magnetizing turns
L_R = mean circumference of ring
I = current in standard solenoid
I_m = magnetizing current

The quantity of electricity Q, expressed in coulombs, discharged through the galvanometer is given by the expression

$$Q = Kd_S = \int idt \tag{32}$$

But

$$i = \frac{e}{R} = \frac{N_S d\phi}{10^8 Rdt} = \frac{N_S A_R}{10^8 R}\frac{dB}{dt} \tag{33}$$

Substituting

$$Kd_S = \frac{N_S A_R}{10^8 R}\int_{-B}^{+B} dB = \frac{2N_S A_R B}{10^8 R} \tag{34}$$

$$B = \frac{KR10^8}{2N_S A_R} d_S. \tag{35}$$

The constant K of the ballistic galvanometer may be obtained by means of the standard solenoid method, as described in Art. **27.** Substituting the value of K from eq. (36), we have

$$B = \frac{4\pi NnA}{10LN_S A_R}\frac{I}{d} \cdot d_S \tag{36}$$

The field strength to which the specimen is subjected is given by the formula

$$H = \frac{4\pi N_m I_m}{10L_R} \tag{37}$$

94. Experiment 15. *Magnetization Curves by the Rowland Ring Method.*—Connect the apparatus as shown in Fig. 61. R is the ring specimen under test, and N_m and N_s the primary and secondary windings, respectively. G is a ballistic galvanometer, and DE a standard solenoid for calibrating it Since a considerable range of current will be required, use two ammeters, one of range 0–15 amperes and the other a millammeter, connected as shown, where S_1 is a knife switch which should be left closed during all manipulations. Open S_1 when it is desired to read the millammeter and then only when the 0–15 ammeter indicates a current less than the full scale reading of the millammeter. First, compute by means of eq. (37), the upper and lower limits of current for fields ranging from .5 to 100 gilberts per centimeter.

Before proceeding to test a specimen, it must first be demagnetized. This is done by applying a magnetizing current somewhat greater than that required for the maximum test field, and reducing it by small steps, reversing the commutator at each step until a current barely readable on the millammeter has been reached. The rate of commutation should not exceed 20 reversals per minute. This should be done with the secondary circuit broken to avoid damaging the galvanometer. Next determine the constant of the galvanometer. To do this, set the double pole double throw switch so as to connect in circuit the primary of the standard solenoid and with a steady current of about 2 amperes, reverse the commutator in such a direction as to cause the galvanometer to swing to high numbers. Make several determinations in this

manner, using such values of current as will give galvanometer deflections ranging from 2 to 5 centimeters. It is necessary here to reverse the primary current, not simply to make or break it, since that is the assumption on which the formula for the galvanometer constant was derived. Use the average value of the ratio of current to deflection in eq. (36).

Everything is now ready for the test proper. Set the double pole double throw switch again so as to include the primary on the ring, and the rheostat R so as to include the maximum resistance. Close the battery circuit and bring the current up to the smallest value computed above. Bring the galvanometer to rest, reverse the primary current, and note the galvanometer deflection. Now bring the commutator back to its original position, increase the current to a slightly greater value, and read the galvanometer deflection again on reversal. Obtain, in this manner, about 15 points on the magnetization curve, spaced more closely together on the lower part of the field strength range where the rise is rapid. Caution.—Succeeding points must always be taken with increasing field strength. Do not allow the current to rise too high and then decrease it. Obtain data for the magnetization curves for two samples of iron. Check your galvanometer constant before and after taking each set.

Report.—1. Plot magnetization curves for the two samples, using B as ordinates and H as abscissas.

2. Compute the permeability for each value of H, and, on a separate sheet, plot permeabilities as ordinates and field strengths as abscissas.

3. For the maximum field strength, compute the magnetomotive force, total flux, and reluctance of the magnetic circuit for each sample, expressing each quantity in its proper units. What is the relation between maxwells and gausses?

95. Experiment 16. *Hysteresis Curves by the Rowland Ring Method.*[1]—Connect the apparatus as indicated in Fig. 61, and observe the precautions regarding use of ammeters, switches, rheostats, etc., indicated in Exp. 15. Instead of starting with zero field and making changes of induction which are symmetrical with respect to the origin, as in the case of the magnetization curve by reversals, start here with the maximum field and make changes of induction by passing first to the retentivity

[1] Ewing, Magnetic Induction in Iron, chap. V.
Taylor, *Physical Review*, vol. 23, p. 95.

point and then away from it. All measurements of induction are accordingly to be made with respect to the retentivity point, and we will, for the moment, regard this point as the origin from which the upper half of the hysteresis curve is to be plotted. The method will be made clear by reference to Fig. 58. Apply first the maximum field, giving $+B_{\max}$ on the curve. Now reduce the field to zero. The induction changes along the upper part of the curve, and goes to the retentivity point, the actual change being equal to B_1. Now apply the field $-H_{\max}$. The induction changes along the curve from B_R to $-B_{\max}$, the actual change in induction being B_2. B_1 and B_2 are thus located on the curve with B_R as the origin. An intermediate point, such as B_3 may be obtained by applying again the field $+H_{\max}$ and slowly reducing to $+H_3$ without breaking the magnetizing current. If the magnetizing current is now broken, the induction again returns to B_R and the change, which is measured by the galvanometer deflection, is B_3. This locates B_3 with respect to B_R. The corresponding point, B_4 may be obtained by applying the field $-H_3$ and observing the throw of the galvanometer. In a similar manner, a series of points, corresponding to pairs of positive and negative values of H, may be obtained and the upper half of the curve plotted with respect to B_R.

The actual manipulation of switches is as follows: Obtain the constant of the galvanometer as explained in Exp. 14. With the galvanometer circuit broken, set the rheostat to give the maximum magnetizing current. Reverse this current several times through the primary coil of the ring to remove the effects of previous magnetization, and thus make sure that the iron will follow the cycle desired. With maximum current flowing, close the secondary circuit and bring the galvanometer to rest. Break the primary circuit by the switch S_2 and observe the throw of the galvanometer which measures B_1. Bring the galvanometer again to rest with the secondary circuit closed. Reverse the commutator. Close S_2 and note the deflection of the galvanometer which measures B_2. Break the secondary circuit, reverse the commutator, bringing the induction back to $+B_{\max}$. Reduce the current, without breaking the circuit, to give a value $+H_3$. Close the secondary circuit and bring the galvanometer to rest. Break the primary by means of S_2 and the throw of the galvanometer measures B_3. Bring the galvanometer to rest, reverse the commutator, and close S_2. The deflection of the galvanome-

ter measures B_4, the induction corresponding to $-H_3$. The other points on the curve are obtained in pairs in the same manner. It is important to notice that before each pair of observations is taken the induction must first be carried to $-B_{\max}$ and then returned to $+B_{\max}$ otherwise a different cycle will be carried out for each pair. Obtain in this way at least ten pairs of values for B, using field strengths ranging from .5 to 30 gilberts per centimeter. It will assist the calculation if deflections corresponding to positive and negative fields are recorded in separate columns. Two samples are to be tested.

The calculation of the values of B is carried out by the same formula as used in Exp. 15 except here we wish the total change in induction instead of half of it as was the case there. Accordingly the limits of integration in eq. (33) are O and B instead of $+B$ and $-B$, giving as our final formula.

$$B = \frac{8\pi NnA}{10LN_sA_r} \cdot \frac{I}{d}(l - f)d_s. \tag{38}$$

Before plotting the curve, the origin should be changed from B_R to O. This is accomplished by adding B_R to all values of induction corresponding to positive fields and subtracting all values of induction corresponding to negative fields from B_R. B_R is determined from the relation

$$B_R = B_{\max} - B_1 = \frac{B_1 + B_2}{2} - B_1 \tag{39}$$

The lower half of the curve, being symmetrical with the upper, is plotted from these same data, merely changing the signs of all values of B.

Report.—1. Plot the hysteresis curves for the two samples of iron, making the plots as large as convenient.

2. Measure the area of the curves by means of a planimeter, and determine the energy loss per cycle per cubic centimeter. Since it is not convenient to plot B and H to the same scale, if unit length along the B axis represents b gausses, and unit length along the H axis, h gilberts per centimeter, unit area will represent $\frac{bh}{4\pi}$ ergs per cc.

3. Compute the Steinmetz coefficient for each sample.

CHAPTER IX

SELF AND MUTUAL INDUCTANCE[1]

96. General Principles.—Whenever a change occurs in the number of magnetic lines linking any electrical circuit, there is induced within the circuit an electromotive force, which, if the circuit is closed, will cause a current to flow. It makes no difference by what means this change is produced; whether magnets in the neighborhood are moved, currents in adjacent circuits changed, or the current in the circuit itself varied, the nature of the induced electromotive force is the same. The direction of the induced electromotive force is given by a simple rule known as Lenz's law, which may be stated as follows: Whenever a change occurs in an electromagnetic system, the direction of the induced electromotive force is such that the magnetic action of its current opposes the change. For example, if the north pole of a magnet is moved toward a closed helix, the induced current flows in such a direction as to produce a north pole on the end toward the magnet, thus tending to repel it, and vice versa, when it is withdrawn. The magnitude of this induced E.M.F. per turn is given by the expression

$$e = \frac{d\phi}{dt} \tag{1}$$

where ϕ is the total flux passing through the turn at any instant.

If the change of flux through the coil is produced, not by moving toward it a magnetic pole but by changing the current in another coil placed near it, the phenomenon of the induced E.M.F. is called mutual induction. The coil which is producing the change of flux is called the primary and that in which the E.M.F. is induced, the secondary. If the current in the primary of two coaxial coils is rising, let us say in the clockwise direction, on looking along the axis, an application of Lenz's law shows that the current in the secondary is flowing counter-clockwise, while if the current in the primary is decreasing, the secondary current

[1] Duff, A Textbook of Physics, p. 445.

Reed and Guthe, College Physics, p. 365. Starling, Electricity and Magnetism, chap. XI.

is in the same direction as the primary. Since the flux through the secondary at any instant is proportional to the current in the primary, we may write for the total E.M.F. in the secondary

$$e = M\frac{di}{dt} \tag{2}$$

where i is the primary current and M a constant depending upon the area of the two coils, their number of turns, distance apart, the permeability of the medium surrounding them, etc. M is called the coefficient of mutual inductance, the unit of which has been named the henry.

Definition.—Two coils have one henry of mutual inductance, if, when the primary current is changing at the rate of one ampere per second, the induced E.M.F. in the secondary is one volt.

When the current through any coil is changing, there is a change of flux, not only through any coil in the neighborhood, but also through the coil itself, causing an induced E.M.F. within it. This phenomenon is known as Self Induction. The direction of this E.M.F., considering the coil to be its own secondary, is determined by Lenz's law, as given above; i.e., when the current is rising, the induced E.M.F. is in such a direction as to oppose the current, and when the current is falling, it tends to maintain it. The induced E.M.F. always opposes any change in the current and is called a counter E.M.F. Since the flux through the coil at any instant is proportional to the current, the induced counter E.M.F. is given by

$$e = L\frac{di}{dt} \tag{3}$$

where i is the current at any instant and L a constant depending upon the number of turns in the coil, its area, shape, permeability of the surrounding medium, etc. L is called the coefficient of self inductance and the unit is the henry.

Definition.—A coil has one henry of self inductance, if, when the current through it is changing at the rate of one ampere per second, the induced counter E.M.F. is one volt.

Since the henry is a relatively large unit, it is customary in expressing the inductance of ordinary coils, to use a unit only one-thousandth as large, called the *millihenry*. Variable standards of self and mutual inductance are made by mounting two coils in such a way that their relative positions, and hence their inductive interactions may be changed. If the coils are

connected in circuit separately, one being used as the primary and the other as the secondary, a calibration curve may be made showing the mutual inductance between them for various positions. If, however, they are connected in series and used as a single coil, a variable self inductance is obtained, since the resultant self inductance of two coils, with mutual inductance between them, is given by the formula

$$L = L_1 + L_2 \pm 2M \tag{4}$$

where L_1 and L_2 are the separate coefficients of self inductance. If the coils are mounted in such a manner that advantage may be taken of both positive and negative values of M, variable self inductances of considerable range may be obtained. Two forms of variable standard are in common use. Figure 62 represents the Ayerton and Perry variable inductor which consists of two coils mounted vertically one of which is fixed and the other movable. The coils are wound on spherical surfaces, and the inner one rotates about a vertical axis. When the planes of the coils are parallel, the resultant self inductance is a maximum or a minimum, according as the mutual is positive or negative. When the coils stand at right angles to each other, the resultant self inductance is the sum of the self inductances of the two coils, since the mutual is zero for this position. For other positions of the movable coil, intermediate values are obtained. The relation between resultant self inductance and angular position is nearly linear. Two pointers on the top read, one the angular position of the coil in degrees, the other the self inductance in millihenries. The coils are joined in series by a flexible conductor. Separate binding posts for the coils are usually provided, and, when used independently, the instrument serves as a variable standard of mutual inductance also.

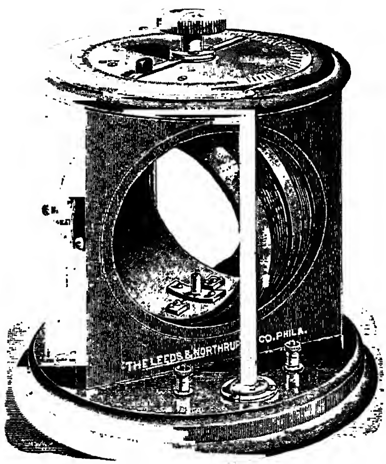

FIG. 62.—Ayerton and Perry variable inductor.

The other instrument is known as the Brook's inductor and is illustrated in Fig. 63. It consists of six coils mounted in pairs in three hard rubber discs, placed one above the other in a horizontal position. The upper and lower disks are fixed and the middle one rotates between them. If the coils are joined in series and connected so that their fields on one side are all directed upward, and on the other side downward, the resultant self inductance is a maximum; but if the middle disk is turned through 180°, the mutual inductance between the fixed and movable coils will neutralize the self inductance and the resultant will be

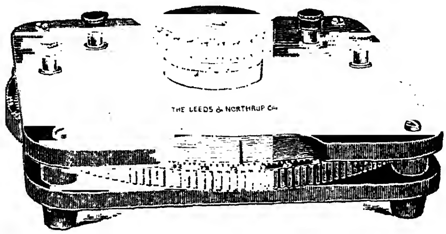

FIG. 63.—Brook's variable inductor.

a minimum. By properly shaping the coils, an approximately linear relation is obtained between angular position and inductance. Separate binding posts enable the coils to be used independently giving also a variable standard of mutual inductance. The instrument is provided with two scales which read respectively self- and mutual inductance in millihenries.

97. Comparison of Inductances.[1]—Two coefficients of self inductance may be compared by a bridge method in which the two coils, whose inductances are to be compared, form two arms of the ordinary Wheatstone bridge. Let L_1 and L_2 of Fig. 64 be two inductances having resistances R_1 and R_2, respectively, and R_3 and R_4 be two non-inductive resistances, and let the bridge be balanced for steady currents, as explained in Art. 35, the condition for which is

$$R_1R_4 = R_2R_3$$

This condition signifies that when the currents, i_1 and i_2 are constant, the potentials at C and D are equal, but less than the

[1] CARHART and PATTERSON, Electrical Measurements, p. 255.
SMITH, Electrical Measurements, p. 197–203.
MAXWELL'S, Elect. and Mag., vol. 2, p. 367.

potential at A. If the battery key K_1 is opened, the current ceases to flow and the potentials at C and D become equal to that at A. When K_1 is again closed, the potentials at C and D on account of the counter E.M.F.'s of self induction in L_1 and L_2 will not necessarily rise at the same rate, although they will come to the same final values. Hence, there may be a short interval of time during which a difference of potential exists between C and D giving a deflection of the galvanometer if K_2 is closed. By properly adjusting L_1 and L_2 it is possible to cause

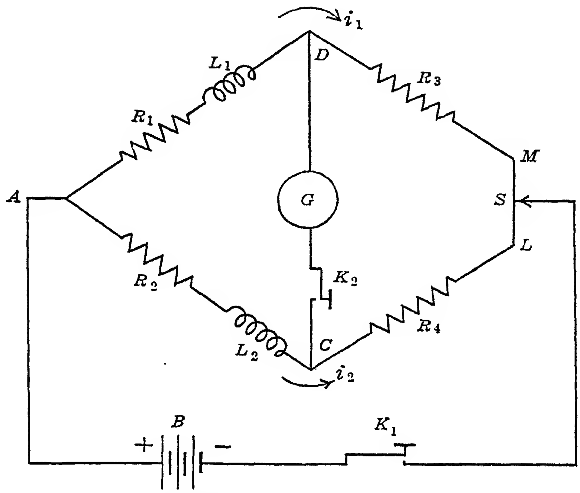

FIG. 64.—Bridge method for self-inductance.

the potentials at C and D to rise at the same rate when the bridge is balanced for both steady and varying currents. The conditions for such a balance is obtained in the ordinary way, except that the equations must include terms representing the fall of potential due to the counter E.M.F. Equating the difference of potential at any instant between A and D to that between A and C, also that between D and S, to that between C and S, we have

$$R_1 i_1 + L_1 \frac{di_1}{dt} = R_2 i_2 + L_2 \frac{di_2}{dt} \tag{5}$$

and

$$R_3 i_1 = R_4 i_2 \tag{6}$$

whence

$$R_3 \frac{di_1}{dt} = R_4 \frac{di_2}{dt} \tag{7}$$

Eliminating i_2 and $\frac{di_2}{dt}$, we have

$$R_4R_1i_1 + R_4L_1\frac{di_1}{dt} = R_2R_3i_1 + R_3L_2\frac{di_1}{dt}. \tag{8}$$

Since $R_1R_4 = R_2R_3$, the condition for steady current balance, the condition for the varying current or inductive balance is

$$L_1R_4 = L_2R_3 \tag{9}$$

or

$$\frac{L_1}{L_2} = \frac{R_3}{R_4} \tag{10}$$

98. Experiment 17. *Comparison of Two Coefficients of Self Inductance by the Bridge Method.*—In carrying out the measurements, it is advantageous to interchange the battery and galvanometer, a procedure which is always possible in any bridge circuit. Connect the apparatus as shown in Fig. 64*A* where L_1 is an unknown inductance and L_2 a variable standard. R_3 and R_4 may be two ordinary resistance boxes with non-inductively wound coils. K_1 and K_2 should be two ordinary press keys. First, using for B a battery of two or three volts, obtain a steady current balance by closing K_1 first, and K_2 after the current has had time to rise to its final value. For the inductive balance, leave the bridge in its state of balance for steady currents and reverse the order of closing the keys. While the current is rising through the inductive arms, the potential at C may be momentarily different from that at D, resulting in a sudden "kick" of the galvanometer, easily distinguishable from the deflection due to unbalance to steady currents. Adjust L_2 until this kick has disappeared. Read the value of L_2 and compute L_1 from equation (10). The unknown inductance to be meas-

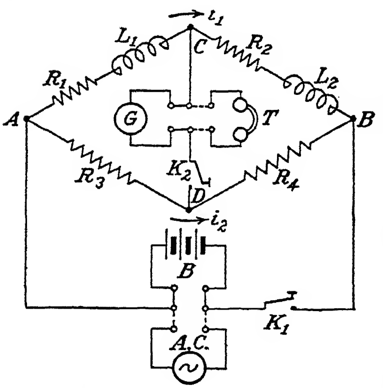

FIG. 64*A*.—Alternate connection for self-inductance.

ured consists of a coil with two independent windings. Determine the inductance of each separately, then join them in series and determine the resultant self-inductances with their mutual inductance aiding and opposing, making in all four measurements.

The inductive balance may be more quickly and accurately obtained by supplying the bridge with alternating current from any of the sources described in Chap. XI, using a pair of head phones to detect the balance point. The double-pole double-throw switches permit a ready interchange of the D.C. and A.C. sources as well as the corresponding devices for detecting a balance.

Note.—It may happen that the balance point lies beyond the range of the variable standard, making an inductive balance impossible. When this happens, the ratio R_3 to R_4 must be changed so as to bring the balance point within the required range. Since a steady current balance must always be obtained first, this requires the insertion of a small non-inductive resistance in series with either R_1 or R_2 as the case may demand. For example, suppose the inductive kick of the galvanometer decreases as L_2 is increased to its maximum, but cannot be made zero or reversed. The combination of the two balance conditions gives

$$\frac{L_1}{L_2} = \frac{R_3}{R_4} = \frac{R_1}{R_2}. \tag{11}$$

If, then, an appropriate resistance is connected in series with R_1 the new steady current balance condition will give a larger ratio of R_3 to R_4 thus making the inductive balance possible. If, on the other hand, L_2 cannot be made small enough, the additional resistance must be placed in series with R_2.

Report.—1. Tabulate your data for the determination of the four inductances as indicated.

2. From the formula $L = L_1 + L_2 \pm 2M$, compute M from the cases where it is aiding and opposing the self inductance. The agreement of these two values gives a check on the accuracy of your work.

3. Show by actual derivation that equation (10) gives the condition for inductive balance for the circuit of Fig. 64*A*.

4. How are coils wound so as to be non-inductive?

99. Measurement of Mutual Inductances.[1]—The mutual inductance of two coils may be measured in terms of capacity and resistance by means of a method due to Carey-Foster, in

[1] Carey-Foster, *Phil. Mag.*, vo. 23, p. 121.
Carhart and Patterson, Electrical Measurements, p. 268.
Smith, Electrical Measurements, p. 217.

which the quantity of electricity induced in the secondary is balanced against a known charge from a standard condenser. The connections are shown in Fig. 65, where P and S are the primary and secondary coils of the mutual inductance to be measured, C a standard condenser, and G a ballistic galvanometer. The primary circuit is represented by the path $BPAR_1$, while the secondary is SR_2DA including the galvanometer G. It will be noted that the galvanometer is also included in circuit DCR_1A containing a standard condenser. When the primary circuit is closed the galvanometer will be traversed by two distinct quantities of electricity: (1) The quantity induced in the secondary coil, and (2) the charge entering the condenser, both of which may

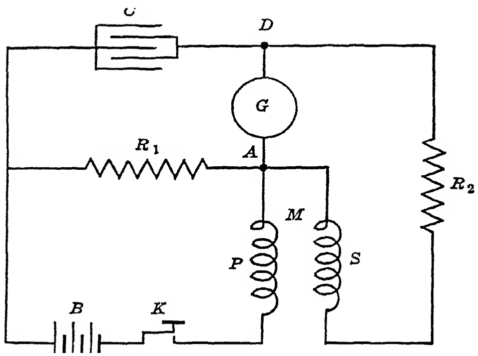

FIG. 65.—Mutual inductance by Carey-Foster method.

easily be computed. If these two quantities are equal and pass through the galvanometer in opposite directions, no deflection will result, which is the balance condition sought.

The quantity Q_1 induced in the secondary coil is the time integral of the secondary current, during the interval required for the primary to rise from zero to its final value I. That is,

$$Q_1 = \int i_2 dt = \frac{M}{R}\int \frac{di_1}{dt}dt \tag{12}$$

$$= \frac{M}{R}\int_0^I di_1 = \frac{MI}{R} \tag{13}$$

where R is the effective resistance of the secondary circuit. The quantity Q_2 of electricity passing through the galvanometer to charge the condenser is given by

$$Q_2 = CV = CR_1 I \tag{14}$$

where $V = R_1I$ is the fall of potential across R_1 which is charging the condenser. Equating,

$$\frac{MI}{R} = CR_1I, \tag{15}$$

or

$$M = CR_1R. \tag{16}$$

Since, at the point of balance, there is no current through the galvanometer, and consequently no fall of potential across it, the effective resistance R of the secondary circuit includes only R_2 and S. The final formula then becomes

$$M = CR_1(R_2 + S) \tag{17}$$

If C is expressed in farads, and the resistances in ohms, M will be given in henries.

100. Experiment 18. *Mutual Inductance by the Carey-Foster Method.*—Connect the apparatus as shown in Fig. 65, where PS is a variable mutual inductance whose calibration curve is to be obtained, C a subdivided standard condenser, G a ballistic galvanometer of long period, and B a storage battery of 20 volts. It is necessary that the four wires indicated at A should actually meet at a common point, so a connector should be used. Since a large voltage is connected directly across R_1 there is danger of burning it, so compute the minimum resistance which may be used, allowing a maximum power consumption of 4 watts per coil. To make sure that the discharges through the galvanometer oppose one another and are of the same order of magnitude, try them first separately; that is, break the circuit at C, make and break the primary circuit and note the direction of the galvanometer deflection at the make, due to the induced current in the secondary. Now close the circuit again at C, breaking the secondary at R_2, and note the deflection at make, which is now due to the charge entering the condenser. If the deflection is in the same direction as before, reverse the connections on either the primary or secondary coil. Close the circuit at R_2 and obtain a balance varying R_1, R_2, and C. The resistance of the secondary coil may be obtained by means of a post-office box.

Report.—1. Plot mutual inductance in millihenries, as ordinates, and positions of coil as abscissas.

2. How would your results be affected if you had interchanged primary and secondary coils? Explain.

CHAPTER X

ELEMENTARY TRANSIENT PHENOMENA[1]

101. Time Constant, Circuit Having Resistance and Inductance.—When an E.M.F. is suddenly impressed on a circuit containing resistance only, the current rises instantly to a definite value determined by Ohm's law. If, however, the circuit contains inductance as well as resistance, this is not the case, for while the current is being established, it produces within the coil a magnetic flux which links the turns of the coil. Whenever a change occurs in the flux through a coil there is induced within it an E.M.F. in such a direction as to oppose the change which produced it. From the definition of self inductance, the value of this counter E.M.F. is $L\frac{di}{dt}$ where L is the coefficient of self inductance. It is thus seen that the impressed E.M.F. is opposed by two counter E.M.F.'s; one due to the current flowing through the resistance and the other due to the rising current in the coil. Such a circuit is represented in Fig. 66, where the resistance R and the inductance L are shown separately, although they may co-exist in the coil. Let the value of the current, t seconds after closing the key, be i. Then by Ohm's law, we have

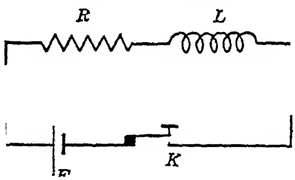

FIG. 66.—Circuit containing resistance and inductance.

$$E \quad Ri + L\frac{di}{dt} \qquad (1)$$

This is a differential equation and can not be solved by the ordinary rules of algebra. Dividing through by R and letting $I = \frac{E}{R}$ be the final value of the current, we have

$$I = i + \frac{Ldi}{Rdt} \qquad (2)$$

[1] BEDELL and CREHORE, Alternating Currents
PIERCE, Electric Oscillations and Electric Waves.
STEINMETZ, Transient Phenomena.

Separating the variables, we obtain

$$\frac{di}{I - i} = \frac{R}{L}\, dt \tag{3}$$

Integration of both sides gives

$$-\log\,(I - i) = \frac{R}{L}\, t + K \tag{4}$$

where K is an arbitrary constant whose value may be obtained by substituting corresponding known values for i and t. Counting time from the instant the key is closed, when $t = o$, $i = o$, and these quantities when substituted in eq. (4) give $K = -\log I$. Hence eq. (4) becomes, on replacing K by its value and rearranging,

$$\log \frac{(I - i)}{I} = -\frac{R}{L} \tag{5}$$

Taking the antilogarithm of both sides,

$$^{-\frac{R}{L}t} \tag{6}$$

where e is the base of the Naperian logarithms. Solving for i, we have

$$i = I\left(1 - e^{-\frac{R}{L}t}\right) \tag{7}$$

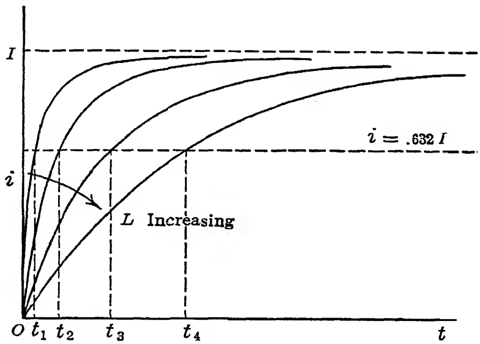

FIG. 67.—Growth of current in a circuit containing resistance and inductance.

The graph of this equation for a series of values of L with constant R and E is shown in Fig. 67. It is seen that when $L = o$ the last term of eq. (7) vanishes and the current rises immediately to its final value; but as L is made larger a longer time is required for it to reach a given fraction of its final magnitude. It is obvious that inductively wound coils might be classified according to the time required for the current to reach a certain specified fraction

of its final values under a constant impressed E.M.F. The most suitable fraction to choose is arrived at in the following way. If, in eq. (7), $t = \frac{L}{R}$, there results

$$i = I\left(1 - \frac{1}{e}\right) = .632\,I$$

The quantity $\frac{L}{R}$ is called the "Time Constant" for the coil and is defined as the time required for the current to reach .632 of its final value under the action of a constant E.M.F. The values t_1, t_2, t_3, etc., in Fig. 67 represent the time constants for the various values of L.

102. Circuit Having Resistance and Capacitance.—A case quite similar to the one discussed above is that in which an E.M.F. is suddenly impressed upon a circuit containing resistance and capacitance in series. Such an arrangement is shown in Fig. 68. As soon as the key is closed, a current flows through R and a charge begins to accumulate in C. This charge at once produces a counter E.M.F., which, added to that due to the current through R, balances the impressed E.M.F.

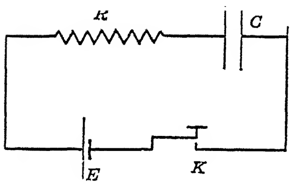

FIG. 68.—Circuit containing resistance and capacitance in series.

The potential difference across the condenser is $\frac{Q}{C}$ or $\frac{1}{C}\int idt$. Accordingly, we may write

$$E = Ri + \frac{1}{C}\int idt. \tag{8}$$

It is more convenient to solve this equation in terms of the instantaneous charge q in the condenser than of the current through the resistance. Remembering that $i = \frac{dq}{dt}$ we have, on substitution in eq. (8),

$$E = R\frac{dq}{dt} + \frac{q}{C} \tag{9}$$

Multiplying through by C and putting $CE = Q$, the final charge in the condenser, eq. (9) becomes, on separating the variables,

$$\frac{dq}{Q - q} = \frac{dt}{RC} \tag{10}$$

Integrating both sides of eq. (10), we have

$$-\log (Q - q) = \frac{t}{RC} + K \tag{11}$$

As before, K is an arbitrary constant of integration which may be evaluated by subtistuting known values of q and t in eq. (11). Counting time from the instant of closing the key, we have, when $t = o$, $q = o$. Substituting in eq. (11)

$$K = -\log Q$$

Replacing K by its value, and rearranging terms, eq. (11) becomes

$$\log \frac{(Q - q)}{Q} = -\frac{t}{RC} \tag{12}$$

Taking the antilogarithm of both sides, we have

$$\frac{Q - q}{Q} = e^{-\frac{t}{RC}} \tag{13}$$

Solving for q, there results

$$q = Q(1 - e^{-\frac{t}{RC}}) \tag{14}$$

This equation is analagous to eq. (7) of the previous article and its graph is shown in Fig. 69, for several values of R with constant

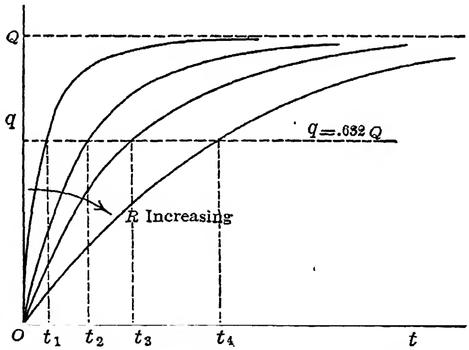

FIG. 69.—Growth of charge for a circuit containing resistance and capacitance.

E and C. If $R = o$, the condenser becomes charged instantly to its final value Q, but when a series resistance is included, a definite time is required for the condenser to become charged. Such circuits may be classified according to the time required for the charge to reach a specified fraction of its final value. As before this fraction is arrived at by putting $t = RC$.

Eq. (14) then becomes

$$q = Q\left(1 - \frac{1}{e}\right) = .632\,Q.$$

The quantity RC is called the time constant for a circuit containing resistance and capacitance, and is defined as the time required for the charge to reach .632 of its final value. These times are shown for the successive values of R by t_1, t_2, t_3, etc., in the figure. The time constant is an important concept in the study of reactive circuits and will be referred to frequently in this text in the discussions to follow.

103. Circuit Containing Resistance, Inductance and Capacitance. Discharge of a Condenser.—To describe some of the phenomena peculiar to a circuit containing resistance, inductance and capacitance, it will be supposed that the parts are connected in series as shown in Fig. 70, and that the condenser has been charged by appropriate means. Suppose further that the key has been closed and that it is discharging; also that the instantaneous current is i and the charge in the condenser is q. Since no external E.M.F. is acting, the sum of the differences of potential across the three elements of the circuit must be zero at all times. Accordingly,

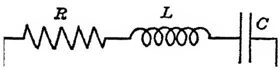

FIG. 70.—Circuit containing resistance, inductance, and capacitance.

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0. \tag{14}$$

Differentiating and dividing through by L we have

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}\,i = 0. \tag{15}$$

Since $i = \frac{dq}{dt}$, eq. (14) may also be written

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}\,q = 0. \tag{16}$$

eqs. (15) and (16) are sufficient to completely describe a circuit of this character. Since they are identical, only one of them, e.g., (15), will be discussed.

This is a linear differential equation of the second order with constant coefficients and may be solved in the following manner: Let

$$i = ke^{mt} \tag{17}$$

where k is an arbitrary constant depending upon the boundary conditions and m, another constant, depending upon the coefficients of the original differential equation. Differentiating eq. (17) twice and substituting in eq. (15), there results

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0. \tag{18}$$

This equation gives the values that must be assigned to m in order that eq. (17) may be the solution of eq. (15). Solving,

$$m = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} \tag{19}$$

It is thus seen that there are two values of m which will make eq. (17) a solution of eq. (15). These give what are known as "particular solutions" and the "complete solution" is obtained by adding them together. Accordingly,

$$i = k_1 e^{-\left[\frac{RC - \sqrt{R^2C^2 - 4LC}}{2LC}\right]t} + k_2 e^{-\left[\frac{RC + \sqrt{R^2C^2 - 4LC}}{2LC}\right]t} \tag{20}$$

The solution for q is identical except that different arbitrary constants will appear. Call them k_3 and k_4. It is to be noted that the coefficient of t in the exponential term contains a radical, the quantity under which may be positive, zero, or negative according to the relative values of R, L, and C. The theory of differential equations shows that the character of the solutions under these circumstances is quite different, and that we have three distinct cases to consider.

Case I. $R^2C^2 > 4LC$. *Non-oscillatory Discharge.*—For simplicity, let

$$\tau_1 = \frac{2LC}{\mathrm{RC} - \sqrt{R^2C^2 - 4LC}} \text{ and } \tau_2 = \frac{2LC}{\mathrm{RC} + \sqrt{R^2C^2 - 4LC}} \tag{21}$$

The solutions of eqs. (15) and (16) may then be written

$$i = k_1 e^{-\frac{t}{\tau_1}} + k_2 e^{-\frac{t}{\tau_2}} \tag{22}$$

$$q = k_3 e^{-\frac{t}{\tau_1}} + k_4 e^{-\frac{t}{\tau_2}} \tag{23}$$

τ_1 and τ_2 are thus seen to be time constants and it is to be noted that when both inductance and capacity are present, the circuit possesses two time constants instead of one as in the cases previously considered. The arbitrary constants k_1, k_2, k_3, k_4 may be determined in the following way. If time is reckoned from the instant the key is closed, then when

$$t = 0,\ i = 0,\ q = Q \tag{24}$$

Substituting these values in eqs. (22) and (23) there results

$$0 = k_1 + k_2 \quad Q = k_3 + k_4 \tag{25}$$

Differentiating eq. (23)

$$i = \frac{dq}{dt} = -\frac{k_3}{\tau_1}e^{-\frac{t}{\tau_1}} - \frac{k_4}{\tau_2}e^{-\frac{t}{\tau_2}} \tag{26}$$

Comparing coefficients in eqs. (22) and (26) we have

$$k_1 = -\frac{k_3}{\tau_1} \text{ and } k_2 = -\frac{k_4}{\tau_2} \tag{27}$$

Substituting the values of k_3 and k_4 from eqs. (27) in (25) and eliminating, the following values are obtained:

$$\begin{aligned} k_1 &= \frac{Q}{\tau_2 - \tau_1} & k_2 &= \frac{Q}{\tau_1 - \tau_2} \\ k_3 &= \frac{Q\tau_1}{\tau_1 - \tau_2} & k_4 &= \frac{Q\tau_2}{\tau_2 - \tau_1} \end{aligned} \tag{28}$$

Substituting these values in eqs. (22) and (23) we have

$$i = \frac{Q}{\tau_2 - \tau_1}\left[e^{-\frac{t}{\tau_1}} - e^{-\frac{t}{\tau_2}}\right] \tag{29}$$

$$q = \frac{Q}{\tau_1 - \tau_2}\left[\tau_1 e^{-\frac{t}{\tau_1}} - \tau_2 e^{-\frac{t}{\tau_2}}\right] \tag{30}$$

It is thus seen that the solutions are made up of two exponential curves whose difference is to be taken. In the case of the current, these curves have the same initial ordinates but approach the time axis at different rates because of the different time constants. The solution is shown graphically in Fig. 71, where the dotted curves are the separate exponentials and the full line represents their difference. The current starts at zero, rises to a maximum and then slowly dies away.

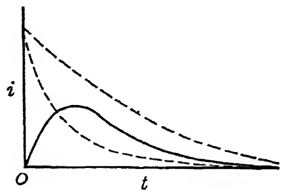

FIG. 71.—Aperiodic discharge of a condenser.

Case II. $R^2C^2 = 4\ LC$. *Critically Damped Discharge.*—In this case the roots of eq. (18) are identical having the value $-\frac{R}{2L}$ and the two terms of eq. (22) are the same. This equation cannot be the complete solution for this case since it contains but one arbitrary constant, whereas the complete solution must have two, since the original differential equation is of the second order.

The theory of differential equations[1] shows that for this case the solutions of eqs. (15) and (16) are

$$i = k_1 e^{-\frac{R}{2L}t} + k_2 t e^{-\frac{R}{2L}t} \tag{31}$$

$$q = k_3 e^{-\frac{R}{2L}t} + k_4 t e^{-\frac{R}{2L}t} \tag{32}$$

Imposing the same boundary conditions as before, namely, when $t = 0$, $i = 0$, and $q = Q$, we have

$$k_1 = 0 \text{ and } k_3 = Q$$

Differentiating eq. (32)

$$i = \frac{dq}{dt} = -\frac{k_3 R}{2L} e^{-\frac{R}{2L}t} + k_4 \left[e^{-\frac{R}{2L}t} - \frac{R}{2L} t e^{-\frac{R}{2L}t} \right] \tag{33}$$

Applying the first boundary condition to eq. (34) gives

$$k_4 = \frac{k_3 R}{2L} = \frac{QR}{2L}$$

Comparison of coefficients in eqs. (31) and (33) gives

$$k_2 = -\frac{R}{2L} k_4 = -\frac{QR^2}{4L^2} = -\frac{E}{L}$$

The complete solutions accordingly are

$$i = -\frac{E}{L} t e^{-\frac{R}{2L}t} \tag{34}$$

$$\left[1 + \frac{Rt}{2L}\right] Q e^{-\frac{R}{2L}t} \tag{35}$$

These equations consist of the product of a straight line and an exponential curve, and are similar to the corresponding ones for Case I. If numerical values are substituted, it is found that they rise to higher values and that they are more compressed along the time axis. In fact, the theory shows that for this critical case the discharge takes place in the shortest time possible.

Case III. $R^2C^2 < 4LC$. *Oscillatory Discharge.*—This is the most interesting and important of the three cases. The quantity under the radical sign of eq. (19) then becomes imaginary and the two roots of eq. (18) are complex quantities. Call them

$$m_1 = \alpha + j\beta \text{ and } m_2 = \alpha - j\beta$$

where

$$\alpha = -\frac{R}{2L}, \ \beta = \frac{\sqrt{4LC - R^2C^2}}{2LC}, \text{ and } j = \sqrt{-1}.$$

[1] Murray. Differential Equations, p. 65.

Equation 20 may then be written

$$\begin{aligned} i &= k_1\, e^{(\alpha + j\beta)t} + k_2\, e^{(\alpha - j\beta)t} \\ &= e^{\alpha t}[k_1 e^{j\beta t} + k_2 e^{-j\beta t}] \\ &= e^{\alpha t}[k_1(\cos \beta t + j \sin \beta t) + k_2 (\cos \beta t - j \sin \beta t)] \\ &= e^{\alpha t}[(k_1 + k_2) \cos \beta t + (k_1 - k_2)\, j \sin \beta t] \end{aligned}$$

Let

$$k_1 = \frac{A - jB}{2} \quad \text{then } k_1 + k_2 = A$$

$$k_2 = \frac{A + jB}{2} \qquad k_1 - k_2 = -jB$$

whence

$$i = e^{\alpha t}[A \cos \beta t + B \sin \beta t]$$

By means of a well known formula of trigonometry this may be written

$$i = k e^{\alpha t} \sin (\beta t + \phi) \tag{36}$$

where

$$k = \sqrt{A^2 + B^2} \text{ and } \phi = \tan^{-1}\frac{A}{B}$$

In a similar manner the solution of eq. (16) for this case may be shown to be

$$q = k' e^{\alpha t} \sin (\beta t + \phi') \tag{37}$$

The four arbitrary constants k, k', ϕ, ϕ' are real quantities and may be determined by imposing the same boundary conditions as used above. Substituting in eqs. (36) and (37) the values $i = 0$, $q = Q$ for $t = 0$ respectively they become

$$0 = k \sin \phi \qquad \text{whence } \phi = 0 \tag{38}$$

$$Q = k' \sin \phi' \qquad \frac{Q}{\sin \phi'}$$

Differentiating eq. (37) with respect to t

$$i = \frac{dq}{dt} = k' e^{\alpha t}[\alpha \sin (\beta t + \phi') + \beta \cos (\beta t + \phi')]$$

$$= k' e^{\alpha t}\left[\sqrt{\alpha^2 + \beta^2} \sin \left(\beta t + \phi' + \tan^{-1}\frac{\beta}{\alpha}\right)\right] \tag{39}$$

Using again the condition $i = 0$ for $t = 0$ we have

$$\tan \phi' = -\frac{\beta}{\alpha} = -\frac{\sqrt{4LC - R^2C^2}}{RC}$$

$$\therefore k' = \frac{Q}{\sin \phi'} \qquad \frac{Q}{\sin \tan^{-1}\dfrac{\sqrt{4LC - R^2C^2}}{RC}} = \frac{\sqrt{4LC}\,Q}{\sqrt{4LC - R^2C^2}}$$

Comparing the coefficients of the sine terms in eqs. (39) and (36) we have

$$k = k'\sqrt{\alpha^2 + \beta^2} = \frac{2Q}{\sqrt{4LC - R^2C^2}}$$

The complete solutions may now be written

$$i = \frac{2Q}{\sqrt{4LC - R^2C^2}} \epsilon^{-\frac{R}{2L}t} \sin \frac{\sqrt{4LC - R^2C^2}}{2LC} t \tag{40}$$

$$\frac{\sqrt{4LC}Q}{\sqrt{4LC - R^2C^2}} \epsilon^{-\frac{R}{2L}t} \sin\left[\frac{\sqrt{4LC - R^2C^2}}{2LC} t + \tan^{-1} \frac{\sqrt{4LC - R^2C^2}}{RC}\right] \tag{41}$$

The current and charge are sine functions of the time and are therefore oscillatory in character. The initial amplitude of the oscillations is proportional to the charge given to the condenser

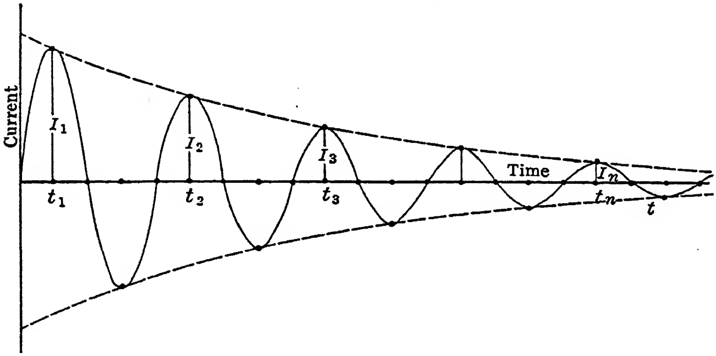

FIG. 72.—Damped sine wave.

and depends also upon the constants R, L, and C of the circuit. Furthermore, the amplitude is multiplied by an exponential factor which decreases with the time and the oscillations consequently die out. An oscillation of this character is spoken of as a "damped" sine wave. The graph for the current wave is shown in Fig. 72. That for the charge is similar to it except that its phase is ahead of the current by the angle whose tangent is given by the last term in eq. (41). If $R = 0$, this angle is 90°.

The period T of the oscillation is obtained from eq. (40) by the relation

$$\omega = \frac{\sqrt{4LC - R^2C^2}}{2LC} = \frac{2\pi}{T}$$

whence

$$T = 2\pi \frac{2LC}{\sqrt{4LC - R^2C^2}}$$

If R^2C^2 may be neglected in comparison to $4LC$, this reduces to the simple expression

$$T = 2\pi\sqrt{LC} \tag{42}$$

104. Logarithmic Decrement.—The physical interpretation of the phenomenon just described in mathematical terms is as follows: When the condenser is given a charge, a definite amount of energy, $\frac{1}{2}C\ V^2$, is stored up in it. As it discharges and current flows through the circuit, this energy is in part dissipated by the resistance R and in part stored up in the electromagnetic field of the inductance L. At the instant the potential difference across the condenser is zero the energy which has not been dissipated as heat is in the coil has the value $\frac{1}{2}LI^2$. This energy, minus that dissipated during the next quarter swing is returned to the condenser charging it in the opposite direction and so on. If the circuit were entirely free from resistance, the oscillations would simply represent interchanges of energy between the condenser and the coil at a frequency twice that of the circuit and would continue indefinitely in much the same manner as a pendulum suspended by frictionless bearings in a vacuum. It is obvious that the greater the rate of energy dissipation, the smaller the number of oscillations. The quantitative method of treating the damping effect is as follows:
Write eq. (40) in the simplified form

$$i = Ie^{-\alpha t} \sin \omega t \tag{43}$$

where

$$I = \frac{2Q}{\sqrt{4LC - R^2C^2}}; \qquad \alpha = \frac{R}{2L}; \qquad \omega = \frac{\sqrt{4LC - R^2C^2}}{2LC}$$

Let $I_1, I_2, I_3 \ldots\ldots I_n$ be the successive current maxima as indicated in Fig. 72, let $t_1, t_2, t_3 \ldots\ldots t_n$ be the times at which they occur, and let T be the period of oscillation. Since

$$\sin \omega t_1 = \sin \omega t_2 = \ldots\ldots = 1$$
$$I_1 = Ie^{-\alpha t_1}$$
$$I_2 = Ie^{-\alpha(t_1 + T)}$$
$$I_3 = Ie^{-\alpha(t_1 + 2T)}$$

$$I_n = Ie^{-\alpha(t_1 + (n-1)T)}$$

The ratio of the first amplitude to any succeeding one is

$$\frac{I_1}{I_n} = e^{\alpha(n-1)T} \tag{44}$$

In particular, let $n = 2$. Then

$$\frac{I_1}{I_2} = e^{\alpha T}$$

It is easily seen that the ratio of any amplitude to the next one succeeding it is constant and is equal to the value just given. Taking the logarithm of both sides

$$\log_e \frac{I_1}{I_2} = \alpha T = \pi R \sqrt{\frac{C}{L}} = \delta \tag{45}$$

The quantity δ is called the "Logarithmic Decrement" and is defined as the Naperian logarithm of the ratio of any amplitude to the next one succeeding it in the same direction, and is given in terms of the constants of the circuit by eq. (45). One of the many applications that may be made of this quantity is the determination of the number of oscillations that the circuit will execute before the amplitude is reduced to an assigned fraction of its initial value. For example, between I_1 and I_{n+1}, there are n oscillations.

Substituting in eq. (44)

$$\frac{I_1}{I_{n+1}} = e^{\alpha n T} = e^{n\delta}$$

or

$$\log_e \frac{I_1}{I_{n+1}} = n\delta$$

whence

$$n = \frac{1}{\delta} \log_e \frac{I_1}{I_{n+1}}$$

where

$\dfrac{I_{n+1}}{I_1}$ is the assigned fraction.

105. Harmonic E.M.F. Acting on a Circuit Containing Resistance, Inductance and Capacitance.—The equation of E.M.F.'s for this case is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = E \sin \omega t. \tag{46}$$

The theory shows that when the second member of a differential equation is different from zero, the complete solution is made up of two parts: (*a*) the solution of the original differential equation when the second member is put equal to zero, and (*b*) the par-

ticular integral. Part (*a*) has already been discussed and it was found to represent a transient phenomenon which quickly dies out. Part (*b*) corresponds to a "forced" oscillation, and represents a steady state. It is the part in which we are interested in problems of continuous alternating currents.

The student familiar with differential equations will remember that equations of the form of (46) are best treated by means of the differential operator "*D*." To apply this, first differentiate eq. (46) with respect to t to remove the sign of integration

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{E\omega}{L}\cos\omega t \tag{47}$$

Introducing the operator D

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)i = \frac{E\omega}{L}\cos\omega t \tag{48}$$

The particular integral which we are seeking is then

$$i = \frac{1}{D^2 + \frac{R}{L}D + \frac{1}{LC}}\frac{E\omega}{L}\cos\omega t \tag{49}$$

The meaning of the "inverse" operator, as the quantity immediately following the equality sign is called, is this: Find a function of i such that when operated on by the coefficient of i in eq. (48) it gives the right-hand member of that equation. There is a well known short method[1] for treating the case of sines or cosines such as eq. (49). It consists simply in expressing the function of D as a function of D^2 and replacing D^2 by minus the square of the coefficient of the independent variable. Accordingly

$$i = \frac{1}{-\omega^2 + \frac{R}{L}D + \frac{1}{LC}}\frac{E\omega}{L}\cos\omega t = \frac{E\omega}{RD + \frac{1}{C} - L\omega^2}\cos\omega t =$$

$$\frac{E\omega\left[RD - \left(\frac{1}{C} - L\omega^2\right)\right]}{R^2D^2 - \left(\frac{1}{C} - L\omega\right)^2}\cos\omega t =$$

$$\frac{E\omega\left[RD - \frac{1}{C} - L\omega^2\right)\right]}{-R^2\omega^2 - \left(\frac{1}{C} - L\omega\right)^2}\cos\omega t$$

[1] See Murray, Differential Equations, p. 77.

$$\frac{E\omega^2 R \sin \omega t}{R^2\omega^2 + \left(\frac{1}{C} - L\omega^2\right)^2} + \frac{E\omega\left(\frac{1}{C} - L\omega^2\right) \cos \omega t}{R^2\omega^2 + \left(\frac{1}{C} - L\omega^2\right)^2}$$

$$\frac{ER \sin \omega t}{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2} + E \cdot \frac{\left(\frac{1}{C\omega} - L\omega\right)}{R^2 + \left(L\omega - \frac{1}{C\omega}\right)} \cos \omega t$$

combining into a single sine function

$$\frac{E}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)}} \sin \left(\omega t - \tan^{-1} \frac{L\omega - \frac{1}{C\omega}}{R}\right) \tag{50}$$

It is thus seen that the current is a sine function and has the same frequency as the impressed E.M.F. In general, it is not in phase with the E.M.F. but lags behind or leads according as $L\omega$ is greater or less than $\frac{1}{C\omega}$. If $L\omega = \frac{1}{C\omega}$, i.e., $\omega = \frac{1}{\sqrt{LC}}$, the current is in phase with the E.M.F. and in this case eq. (50) becomes

$$i = \frac{E}{R} \sin \omega t$$

which is identical with the current equation given directly by Ohm's law for the case when no inductance or capacitance is present. The maximum value of the current is obtained by putting the sine function equal to unity: *i.e.*,

$$I = \frac{E}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}$$

By analogy with Ohm's law the denominator is called the "Impedance" of the circuit and the quantities $L\omega$ and $\frac{1}{C\omega}$ are called the inductive and capacitive "Reactances" respectively. Reactance produces not only a phase angle between the current and E.M.F. but also reduces the magnitude of the current.

106. Alternative Method.—For those unfamiliar with differential equations, an indirect method of obtaining the solution of eq. (46) may be employed. Since an alternating E.M.F. is applied to the circuit, it is reasonable to suppose that the current will also be alternating, that it will have the same frequency as the E.M.F.

and that it may not be in phase with the E.M.F. These assumptions are combined in the following expression

$$i = I \sin (\omega t + \phi) \tag{51}$$

where I and ϕ are arbitrary constants which are to be determined by substituting eq. (51) in (46) and finding the values which must be assigned to them in order that eq. (46) may be satisfied.

$$\frac{di}{dt} = I\omega \cos (\omega t + \phi) \text{ and } \int idt = -\frac{I}{\omega} \cos (\omega t + \phi)$$

Substituting these values, eq. (46) becomes

$$LI\omega \cos (\omega t + \phi) + RI \sin (\omega t + \phi) - \frac{I}{C\omega} \cos (\omega t + \phi) = E \sin \omega t$$

Since this equation holds for all values of t, we may write, when

$$\omega t + \phi = 0, \quad LI\omega - \frac{I}{C\omega} = -E \sin \phi$$

when $$\omega t + \phi = \frac{\pi}{2}, \quad RI = E \cos \phi$$

Squaring and adding the above expressions,

$$R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2\Big] I^2 = E^2$$

$$\therefore I = \frac{E}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}$$

Dividing one by the other

$$-\tan \phi = \frac{L\omega - \frac{1}{C\omega}}{R} \text{ or } \phi = -\tan^{-1} \frac{L\omega - \frac{1}{C\omega}}{R}$$

Substituting these values in eq. (51) we have

$$i = \frac{E}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}} \sin \left[\omega t - \tan^{-1} \frac{L\omega - \frac{1}{C\omega}}{R}\right]$$

which is eq. (50) above.

107. Vector Diagrams.—In the discussion thus far, we have spoken of alternating E.M.F.'s and alternating currents and have used in each case the trigonometric expressions in discussing them. For example the equations

$$e = E \sin \omega t$$
$$i = I \sin (\omega t - \phi)$$

have been used to represent respectively an alternating E.M.F. having a maximum value E and a frequency $f = \frac{\omega}{2\pi}$, and an alternating current of the same frequency with a maximum value I lagging behind the E.M.F. by a phase angle ϕ.

These may be regarded as being given by the projections on the Y axis of the vectors OE and OI respectively of Fig. 73 which rotate with constant angular velocity in counter clockwise direction, the latter lagging behind the former by the angle ϕ. The vectors OE and OI represent the maximum values of the E.M.F.

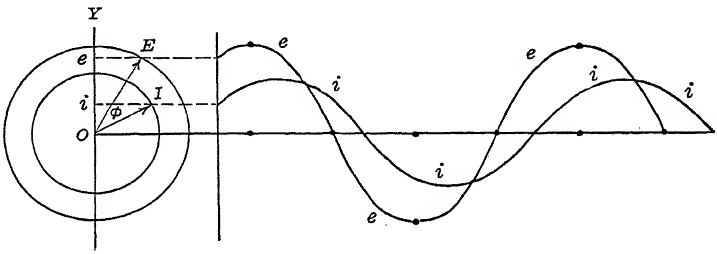

FIG. 73.—Sine waves represented by rotating vectors.

and current. As a special case, consider that of an alternating E.M.F. acting on a circuit having resistance and inductance. The current is given by eq. (50) with $C = \infty$, the condition for zero capacitive reactance. Thus

$$i = \frac{E}{\qquad\qquad} \sin\left(\omega t - \tan^{-1}\frac{L\omega}{R}\right) \tag{52}$$

For the maximum value of the current, we have

$$I = \frac{E}{\sqrt{R^2 + L^2\omega^2}} \text{ or } E = \sqrt{R^2I^2 + L^2\omega^2I^2}$$

From the form of the latter expression it is evident that E has such a value that it may be given as the diagonal of a rectangle whose sides are RI and $L\omega I$, as shown in Fig. 74. The current I is represented as a vector in phase with RI, since, from eq. (52), the current lags behind the E.M.F. by an angle whose tangent is $\frac{L\omega}{R}$. This is the angle ϕ shown in the figure. If this figure is rotated about the origin O with an angular velocity ω in the positive direction, the projections of the vectors, E, RI, and $L\omega I$ upon the Y axis give the instantaneous values of the impressed E.M.F.

and the E.M.F. across the resistance and the inductance respectively.

In a similar manner, a vector diagram may be constructed for a circuit containing resistance, inductance and capacitance in series. For this case the maximum current and phase angle are given respectively by

$$I = \frac{E}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}$$

$$\phi = \tan^{-1}\frac{L\omega - \frac{1}{C\omega}}{R}$$

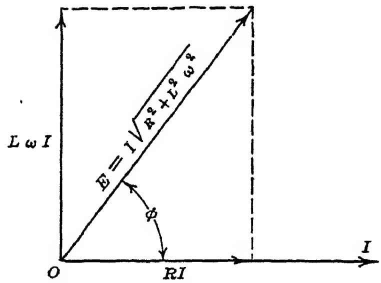

FIG. 74.—Vector diagram for resistance and inductance.

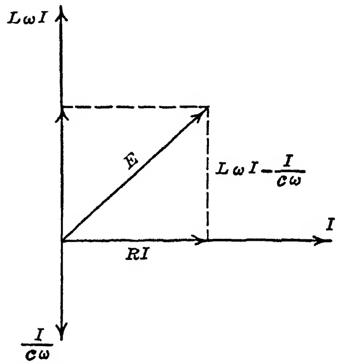

FIG. 75.—Vector diagram for resistance, inductance and capacitance.

The vectors are similar to those of the previous case except for the additional vector, $\frac{I}{C\omega}$, which is shown drawn downward in Fig. 75, since in the equation it appears as a quantity subtracted from $L\omega$. In the figure, $L\omega I$ is shown greater than $\frac{I}{C\omega}$ and the current lags behind the E.M.F. If $L\omega = \frac{1}{C\omega}$, the component of E perpendicular to I is zero and the current is in phase with the E.M.F. On the other hand when $\frac{1}{C\omega}$ is greater than $L\omega$, ϕ is negative, and the current leads the E.M.F.

108. Electrical Resonance.—In discussing the discharge of a condenser through a circuit containing resistance and inductance, it was shown that when the resistance is less than a certain critical

value, oscillations occur. If such a circuit is acted upon by an alternating E.M.F. whose frequency is the same as the natural frequency of the circuit, alternating currents of large amplitude are set up in the inductance and condenser. This phenomenon is spoken of as electrical resonance and is analogous to the motion of a mechanical system possessing inertia and elasticity, when acted upon by an alternating mechanical force having a frequency corresponding to its own free period. Two distinct cases occur depending upon whether the inductance and capacitance are in series with the E.M.F. or are connected across it in parallel. These are distinguished as "Series Resonance" and "Parallel Resonance" respectively.

109. Series Resonance.—This case has been discussed above in some detail. The instantaneous value of the current must satisfy eq. (46) the solution of which is eq. (50). The amplitude of the current is given by

$$I = \frac{E}{\sqrt{R^2 + \left(L\omega - \dfrac{1}{C\omega}\right)^2}}$$

and it has already been pointed out this is a maximum when $L\omega = \frac{1}{C\omega}$, which is the condition for resonance. The current is then given by E divided by R as required by Ohm's law. The resonance condition depends upon the relative values of L, C and ω,

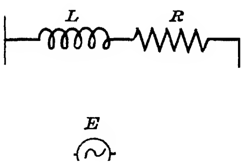

FIG. 76.—Series resonance.

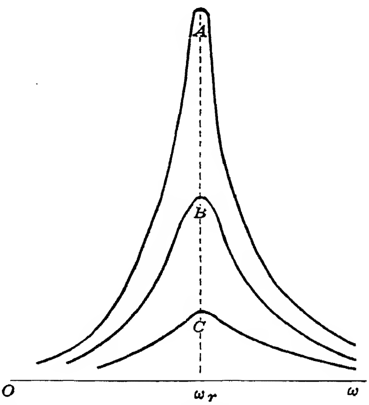

FIG. 77.—Effect of resistance on sharpness of resonance.

and may be brought about by a suitable change of any one of them, the other two being held constant. Bringing a circuit into resonance is generally spoken of as "tuning" it.

The dependence of the current upon the constants of the

circuit may be illustrated by the curves shown in Fig. 77, where the current amplitude is shown as a function of frequency for a short range each side of resonance. The inductance and capacitance are held constant and three different resistances are indicated. A represents the current at resonance for a small resistance and C that for a large. It is to be noted that the effect of a change in resistance is much more marked at resonance than at a frequency somewhat removed. This is because at resonance, resistance alone determines the current, while at low frequencies, the capacity reactance $\frac{1}{C\omega}$ is an important term, but at high frequencies, the inductive reactance $L\omega$ becomes effective in reducing the current. It is to be noted also, that for low frequencies the current leads the E.M.F., is in phase with it at resonance and lags behind at high frequencies. When the resistance is small, the rate of change of the phase angle in passing through resonance is rapid.

110. Parallel Resonance.[1]—When the E.M.F. is introduced in the circuit in such a way that the inductance and condenser are in parallel, the phenomena are strikingly different from those of the series arrangement just described. The connections for this case are shown in Fig. 78. Assuming that the condenser is free from energy absorption, the current through it leads the E.M.F. by ninety degrees, while that through the inductance lags behind by an angle depending upon R, L, and ω. The current in the main circuit is the vector sum of these two and in determining it the relative phases of the components must be taken into account. Denoting the currents through the inductance and condenser by I_L and I_C respectively, their amplitudes are obtained from eq. (50) as follows

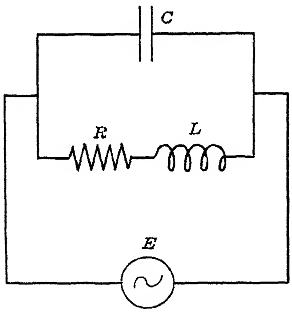

Fig. 78.—Parallel resonance.

$$I_L = \frac{E}{\sqrt{R^2 + L^2\omega^2}} \qquad I_C = EC\omega \tag{53}$$

Letting the vector OE of Fig. 79 represent the impressed E.M.F.,

[1] Circ. 74. U. S. Bureau of Standards, p. 39.

the above currents are given by OI_C and OI_L respectively, and the resultant current OI, is the diagonal of the parallelogram formed by them as sides. The amplitude of the resultant current, by the law of cosines, is:

$$I^2 = I_L^2 + I_C^2 - 2I_LI_C \cos \psi \qquad (54)$$

The value of $\cos \psi$ may be obtained by remembering that the E.M.F. across the coil is made up of two parts: That across the resistance, RI_L, and that across the inductance $L\omega I_L$. The former is in phase with I_L and the latter, ninety degrees ahead of it. Accordingly

$$\cos \psi \quad \frac{L\omega I_L}{E} \qquad (55)$$

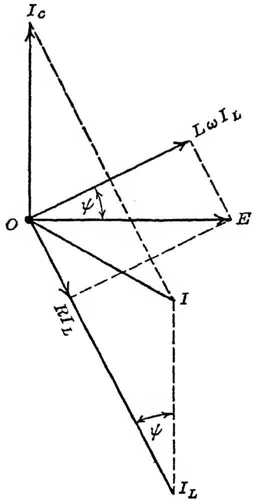

FIG. 79.—Vector diagram for parallel resonance.

Substituting eqs. (53) and (55) in (54) and combining we have

$$I^2 = E^2\left[C^2\omega^2 + \frac{1}{R^2 + L^2\omega^2} - \frac{2C\omega L\omega}{R^2 + L^2\omega^2}\right] \qquad (56)$$

Multiplying numerator and denominator of the second term by $R^2 + L^2\omega^2$, eq. (56) may be written

$$I = E\sqrt{\left(C\omega - \frac{L\omega}{R^2 + L^2\omega^2}\right)^2 + \frac{R^2}{(R^2 + L^2\omega^2)^2}} \qquad (57)$$

Equation (57) is the general expression for the current drawn from the supply, and may either lead the E.M.F. or lag behind it. In Fig. 79, I lags behind E. If the inductance, capacitance, and resistance are properly chosen, the vector OI may be coincident with OE, indicating that the current is in phase with the E.M.F. and the circuit behaves as a pure resistance. In this case OII_L is a right-angled triangle, and the condition to be satisfied is

$$I^2 = I_L^2 - I_C^2$$

Substituting the values of these currents from eqs. (53) and (56) and simplifying we have, as the condition for zero phase angle,

$$C\omega = \frac{L\omega}{R^2 + L^2\omega^2} \qquad (58)$$

The value of ω obtained from this equation is not exactly that corresponding to the natural period of the circuit but approxi-

mates it closely. If R is zero, it corresponds exactly. Imposing this condition on the general expression eq. (57) there results

$$I = \frac{ER}{R^2 + L^2\omega^2} = \frac{E}{\frac{L}{CR}} \tag{59}$$

It is important to note that for small values of R, I is nearly proportional to R and that if R were zero, I would also be zero. We thus have the extraordinary situation in which the larger the resistance the larger the current. Figure 80 shows the variation of current with frequency in the neighborhood of resonance. It is interesting to note that in the case of series resonance the individual voltages across the condenser and coil exceed the total voltage across the two combined, while in parallel resonance, the current in each exceeds the two combined. The series arrangement gives a low impedance at resonance, while the parallel connection gives a high impedance at this point. For this reason, the latter is frequently inserted in a circuit when it is desired to suppress a particular frequency in a complex wave.

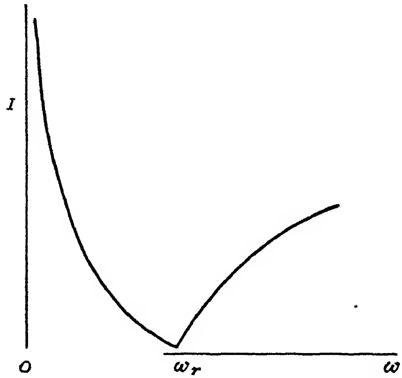

FIG. 80.—Dependence of current on frequency for parallel resonance

111. Measurement of Inductance and Capacitance by Resonance.—The phenomenon of electrical resonance furnishes a convenient method for the determination of inductance and capacitance, particularly when they are small. If two circuits, adjusted to have the same natural periods are placed in inductive relation, and one of them is caused to oscillate, the other will oscillate also by resonance. It was shown on page 136 that the period of an oscillating circuit is given by the expression

$$T = 2\pi\sqrt{LC}.$$

Consequently, the condition for resonance is that the LC products for the two circuits must be the same or

$$L_1C_1 = L_2C_2 \tag{60}$$

where the subscripts refer to the circuits 1 and 2 respectively.

If three of these quantities or one LC product and either L or C are known, the fourth may be computed. In carrying out the measurement it is more satisfactory to use a third circuit as a source of oscillations, and then adjust both the standard and unknown circuits to resonate to it. The inductance and capacitance of the third circuit should be adjustable, but need not be known. The three circuits are shown in Fig. 81. The source circuit is energized by means of the battery A and an ordinary buzzer B which serves as an interrupter. When the armature of the buzzer closes the circuit, the battery current flows through the coil L_3 and stores up energy in the electromagnetic field linking its windings. When the armature of the buzzer breaks the battery circuit, this energy is transferred back and forth between C_3 and L_3 until it has been dissipated. A group of damped oscil-

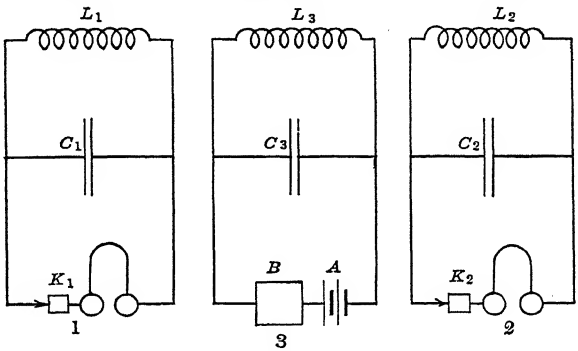

Fig. 81.—Circuits arranged for electrical resonance.

lations is thus established in this circuit for each vibration of the buzzer armature. Similar oscillations but of weaker intensities will be set up in circuits 1 and 2 if they are adjusted to resonate to 3.

If small inductances and capacitances are used the frequency of the oscillations thus produced will be above the audible range, and special means for detecting them must be employed. A convenient method is to use a pair of head phones and a crystal detector such as is commonly employed for the reception of radio signals. Because of the rectifying action of the point-crystal contact the high frequency alternating voltage across the condenser will produce a series of high frequency unidirectional pulses in the phone circuit. Because of the distributed capacitance of the phone windings, these are smoothed out into a single

pulse which causes a vibration of the diaphragm. The sound in the phones then has the period of the buzzer armature. If a sufficient amount of energy is available, it is best to disconnect the right-hand phone lead shown in the figure, and use only a single wire from the phone through the crystal to the oscillatory circuit. This is particularly important when the condensers are small since the capacity between phone leads may introduce a very appreciable error.

112. Experiment 19. *Measurement of Inductance and Capacitance by Resonance.*—Connect the apparatus as shown in Fig. 81, using for L_1 a standard inductance variable by steps, and for C_1 a variable standard air condenser. L_2 should be a single layer coil of uniform windings whose dimensions may easily be measured. C_2 should be an air condenser with plates easily accessible for measurement. First obtain resonance in circuit 2 by varying the frequency of the source. Next obtain resonance in circuit 1 and compute the LC product. Measure the dimensions of C_2 and compute its capacity from eq. (19) given on page 94. (See also the Appendix.) Determine L_1 from eq. (60). Check your result by computing the inductance of L_1 from dimensions using the formula given in the Appendix.

113. Effective Value of an Alternating Current.—If an alternating current is passed through an ordinary D.C. ammeter, no indication will be registered, since such an instrument indicates average values, which in this case is zero. However, if an alternating current is passed through a resistance, heat is liberated, the energy of which is furnished by the current. The reason for the difference in effect in these two cases is that the torque on the moving coil of the instrument is proportional to the current and therefore reverses sign with it, while the heating effect of a current is proportional to its square and is therefore positive no matter what its direction.

It is customary to define the *Effective* value of an alternating current as the equivalent direct current which liberates the same amount of heat in a given resistance per unit time. In deducing the relation between the effective value of an alternating current and its amplitude or maximum value, it is sufficient to equate the heat, in joules, developed by each during the time T of one complete cycle. Accordingly let $i = I \sin \omega t$ be the alternating current and I_e its effective value. When flowing through a resistance R, the heat liberated by each is

$$H = I_e^2RT = \int_o^T i^2Rdt = I^2R\int_o^T \sin^2 \omega t dt \qquad (61)$$

$$= I^2R\int_o^T\left(\frac{1}{2} - \frac{\cos 2\omega t}{2}\right)dt$$

$$= I^2R\left(\frac{t}{2} - \frac{\sin 2\omega t}{4\omega}\right)\Big]_o^T \qquad I^2RT$$

Therefore:

$$I_e = \sqrt{2} = .707I \qquad (62)$$

The above process is seen to be equivalent to squaring the instantaneous values of the current, taking the average value of the squares, and then extracting the square root. The

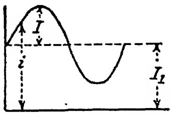

Fig. 81A.—Direct and alternating currents combined.

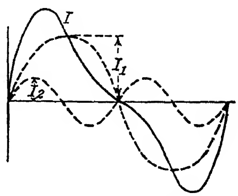

Fig. 81B.—Superposition of two alternating currents.

effective value accordingly is often spoken of as the "Root Mean Square" value. The same considerations hold for an alternating E.M.F.

113A. Effective Value of an Alternating and a Direct Current Combined.—It frequently happens that an alternating current is superimposed upon a direct current, as for example, in the plate circuit of a vacuum tube used as an amplifier or oscillator. As above, the effective value of the combined currents is the equivalent direct current which liberates the same amount of heat in a resistance R. Let

I = amplitude of alternating component.
I_1 = direct current component.
I_e = effective value of the currents combined.
$i = I_1 + I \sin \omega t$ = instantaneous total current.

The total heat liberated per cycle is given by

$$H = I_e^2RT = \int_o^T i^2Rdt = \int_o^T (I_1 + I \sin \omega t)^2Rdt$$

$$= R\int_o I_1^2 + 2II_1 \sin \omega t + I^2 \sin^2 \omega t)dt \qquad (63)$$

Carrying out the integration, the second term vanishes, while the third is the one considered in the preceding article. Thus we have, dividing out R and T

$$I_e^2 = I_1^2 + \frac{I^2}{2} = I_1^2 + (I_e)^2_{AC} \tag{64}$$

where $(I_e)_{AC}$ is the effective value of the alternating component or

$$I_e = \sqrt{I_1^2 + (I_e)^2_{AC}} \tag{65}$$

Thus if, in a circuit carrying a combined direct and alternating current, there are connected in series a direct-current meter of the permanent-magnet type, an induction type A.C. meter unaffected by direct currents, and a hot-wire meter, reading effective values, the reading of the latter will be the square root of the sum of the squares of the other two.

113B. Effective Value of Two Alternating Currents of Different Frequencies.—Let two alternating currents of amplitudes I_1 and I_2, as shown in Fig. 81B be flowing simultaneously through a resistance R and, for simplicity, suppose their frequencies to be ω and 2ω respectively, and let I_e be their effective value combined. The heat liberated in R in the time T, the period of the current of lower frequency, is

$$H = I_e^2RT = \int_o^T i^2Rdt \tag{66}$$

$$= R\int_o^T (I_1 \sin \omega t + I_2 \sin 2\omega t)^2 dt$$

$$= R\int_o^{T} (I_1^2 \sin^2 \omega t + I_2^2 \sin^2 2\omega t + 2I_1I_2 \sin \omega t \sin 2\omega t)dt \tag{67}$$

The first two integrals have been considered in Art. 113 and the third may be evaluated as follows:

$$\int_o^T \sin \omega t \sin 2\omega t dt = 2\int_o^T \sin^{-2} \omega t \cos \omega t dt = \frac{2}{3\omega} \sin^{-3} \qquad = 0$$

Substituting these values for the integrals and dividing out R and T, we have

$$I_e^2 = \frac{I_1^2}{2} + \frac{I_2^2}{2} = I_{e1}^2 + I_{e2}^2$$

or

$$I_e = \sqrt{I_{e1}^2 + I_{e2}^2} \tag{68}$$

Where I_{e1} and I_{e2} are the effective, i.e., root mean square values, of the individual alternating currents.

114. Power Consumed by a Circuit Traversed by an Alternating Current.—Let us suppose that an alternating E.M.F. is impressed upon a circuit which contains reactance as well as resistance so that the current and E.M.F. are not in phase. It is desired to find the power consumed by the circuit. Let the E.M.F. and current be given respectively by the following expressions

$$e = E \sin \omega t; \; i = I \sin(\omega t \pm \phi) \tag{69}$$

where ϕ is the angle of lag or lead.
The energy dH consumed in the time dt is

$$dH = eidt = EI \sin \omega t \sin(\omega t \pm \phi)dt \tag{70}$$

The energy H consumed per cycle is

$$\begin{aligned} H &= EI \int_o^T \sin \omega t(\sin \omega t \cos \phi \pm \cos \omega t \sin \phi)dt \\ &= EI \left\{ \cos \phi \int_o^T \sin^2 \omega t dt \pm \sin \phi \int_o^T \sin \omega t \cos \omega t dt \right. \\ &= EI \quad \cos \phi \int_o^T \left(\frac{1}{2} - \frac{\cos 2\omega t}{2}\right)dt \pm \sin \phi \int_o^T \sin \omega t \cos \omega t dt \end{aligned}$$

Carrying out the integrations and substituting limits the last two integrals vanish and we have

$$H = EI \frac{T}{2} \cos \phi$$

$$\text{Power} = \frac{\text{Energy per Cycle}}{T} = \frac{E}{\sqrt{2}} \frac{I}{\sqrt{2}} \cos \phi$$

It is thus seen that the power consumed is the product of the effective E.M.F. and current multiplied by the cosine of the phase angle. Cosine ϕ is called the "Power Factor" and varies from zero to unity.

The physical significance of the power factor can readily be seen in the simple case of a circuit containing inductance and resistance only. Referring to Fig. 74, page 142, we see that the angle φ of this figure is, as was stated, the phase angle. From this figure it can readily be seen that

$$E \cos \varphi = RI$$

Substituting this value in the expression for the power above, we see that

$$\frac{E}{\sqrt{2}} \frac{I}{\sqrt{2}} \cos \varphi = \frac{R I I}{}$$

or

$$\text{Power} = RI_e^2$$

Thus in this circuit the only power loss is in the heating of the resistance. In more complicated circuits, such as where power is transformed into mechanical work, the interpretation is more complicated.

CHAPTER XI

SOURCES OF E.M.F. AND DETECTING DEVICES FOR BRIDGE METHODS

Before discussing the various bridges which are to be employed in the measurement of inductance and capacitance, the student should become familiar with some of the sources of alternating E.M.F. and detecting devices that are available. Inasmuch as the alternating currents for commercial purposes are of frequencies too low to give a tone suitable for telephonic balances, special generators have been devised, a few of which will now be described.

115. The Sechometer.—In Exps. 12 and 17 methods were employed for comparisons of capacitance and inductance respectively in which batteries were employed to energize the bridges and the E.M.F.'s due to the reactances were made manifest during the rise and fall of the bridge currents following the closing and opening of the battery circuit key. It was then found that the galvanometer deflected in one direction on closing and in the opposite on opening this key. If some means were available by which the galvanometer leads could be interchanged each time the key is opened and closed, the deflections would always be in the same direction and if the interval between successive operations of the key were small compared to the period of the galvanometer, a steady deflection would result whereby the sharpness of the bridge balance would be greatly increased. The Sechometer is a device which accomplishes this purpose and derives its name from the "Secohm" by which our present unit of inductance, the henry, formerly was known.

It consists essentially of two commutators mounted on the same shaft which may be driven at any desired speed by a motor. The segments are set in such a relation to each other that the galvanometer leads are interchanged by one commutator each time the polarity of the battery is reversed by the other. The connections are shown in Fig. 82. The device must not be driven at too high a speed since sufficient time must be allowed for the establishment of a steady state at each reversal. High speeds

also develop heat at the brush contacts resulting in errors due to thermal E.M.F.'s.

It is well to get an approximate bridge balance by manipulating

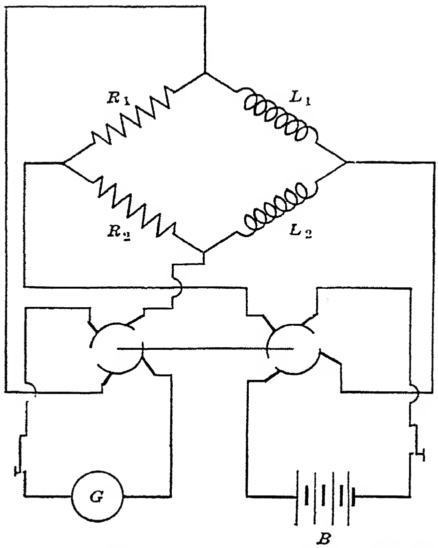

FIG. 82.—Sechometer connections to bridge.

the battery and galvanometer keys with the sechometer stationary and then use it merely to obtain the final setting. The

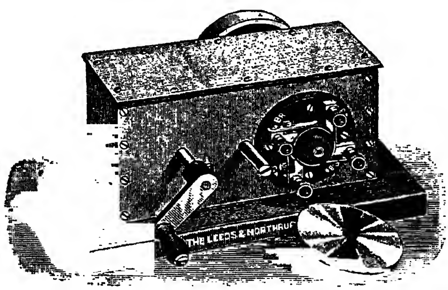

FIG. 83.—The Sechometer.

application of this instrument is equivalent to using a generator giving a square wave form and can, therefore, be used only with bridges which balance independent of the frequency. Figure 83

shows the assembled instrument provided with a crank for hand driving.

116. The Wire Interrupter.—The vibrating wire interrupter, shown in Fig. 84, consists essentially of a piano wire stretched between rigid supports A and B, the tension of which may be varied by the screw S. Vibrations are maintained by means of an electromagnet M, intermittently energized by a battery B_1. The mercury cup contact C_1 interrupts this current when the wire is drawn up, and the device operates in a manner similar to the ordinary buzzer. The battery B_2, which supplies current to the

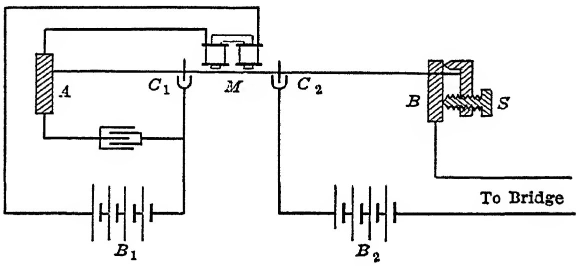

FIG. 84.—Vibrating wire.

bridge, is connected through the contact C_2 and this circuit is also closed and opened at the frequency maintained by the wire.

Frequencies ranging from 25 to 150 are easily secured. This device is particularly well suited for use with the vibration galvanometer, since it permits of sharp, easy tuning, and may readily be adjusted to resonance with the galvanometer. Since the vibration galvanometer responds only to the fundamental and not the overtones, the fact that the interrupter gives a square wave form results in no disadvantage and the combination may accordingly be used on bridge circuits which do not balance independent of the frequency and the same results obtained as though a source giving a pure sine wave were employed. If a suitable condenser K is shunted across the contact C_1 to prevent arcing, the device will operate continuously for hours with little or no attention.

117. The Motor Generator.—Another inexpensive source of alternating current is the small motor generator set manufactured by the Leeds and Northrup Company shown in Fig. 85. The generator, which is shown at the right in the figure, is of the

inductor type and has stationary windings for both field and armature circuits. Direct current is supplied to the field coil at the base thus energizing the magnetic circuit which includes the broad toothed wheel carried on the armature shaft of the motor. The reluctance of this magnetic circuit depends upon the position of the teeth with respect to the pole pieces. When the wheel is driven, the flux through the magnetic circuit fluctuates at a frequency equal to the number of teeth passing the pole tips per second. An alternating E.M.F. is thus induced in the armature windings placed near the pole tips.

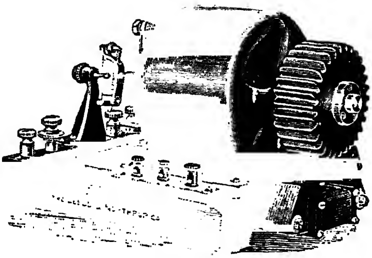

FIG. 85.—Motor generator set.

By properly choosing the shape of the pole tips and teeth, it is possible to obtain a wave form that is relatively free from harmonics although it is not possible to eliminate them completely. By the use of a suitable filter, good wave forms may be obtained. By means of a specially designed mechanical governor, the speed of the motor may be maintained constant to ½ per cent.

118. The Microphone Hummer.—If it is not necessary to supply the bridge with a constant frequency, a simple microphone hummer furnishes a convenient and inexpensive source of alternating current. Such a circuit is shown in Fig. 86. It consists of a microphone transmitter facing a telephone receiver. The transmitter and receiver circuits are connected in the ordinary way by a telephone transformer. Any stray sound will cause a variation in the microphone current which produces a sound in the receiver. This sound is "fed back" to the microphone which again produces a sound in the receiver and the action is thus continuous, the energy being supplied by the battery. The interval between the application of the sound to the microphone and its return by the receiver after having operated the electrical circuits depends to a large extent upon the time constant of the

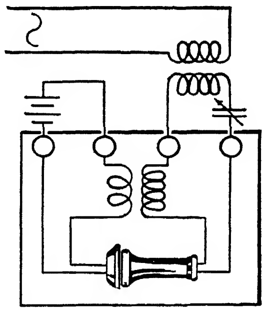

FIG. 86.—Microphone hummer.

secondary or receiver circuit. If the resistance of this circuit is not too large, it may be made oscillatory by the introduction of a condenser as indicated in the diagram. By giving suitable values to the capacitance of this condenser a large range of frequencies may be obtained. Another transformer, the primary of which is in series with the receiver, furnishes a means of making the A.C. power thus generated available for a bridge circuit. A convenient form of this device is manufactured by R. W. Paul of London under the trade name "Kumagen," the appropriateness of which is easily understood. The microphone, receiver, and transformers are contained in a felt lined case which serves to deaden the sound. A condenser is also furnished which is variable in steps chosen so as to give a number of suitable frequencies.

119. The Audio-oscillator.—The frequency of the microphone hummer, described above, is somewhat variable depending upon

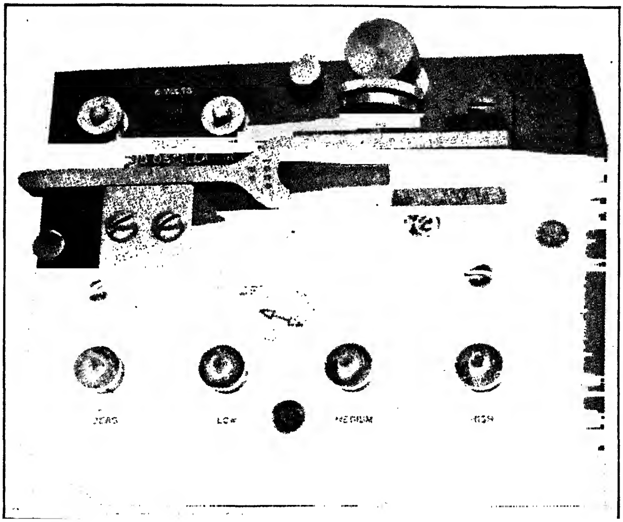

FIG. 87.—Audio-oscillator.

the strength of the driving battery and the load upon the secondary of the output transformer. An adaptation of the underlying principle has been made by Campbell by which this objection is overcome. It consists in operating the microphone button, not by sound waves from a telephone receiver, but by means of a tuning fork whose mechanical period coincides with

the period of the electrical circuit which it energizes. Several different forms are on the market. Figure 87 shows an instrument of this type known as the audio-oscillator, manufactured by the General Radio Co., and Fig. 88 gives the wiring diagram. The "field coil" which is connected directly across the battery serves merely to magnetize the fork and armature core to a point on the magnetization curve near the maximum permeability and this increases the attractive forces of the poles. The battery also sends current through the microphone and primary of the input transformer. When the battery key is closed, the current through

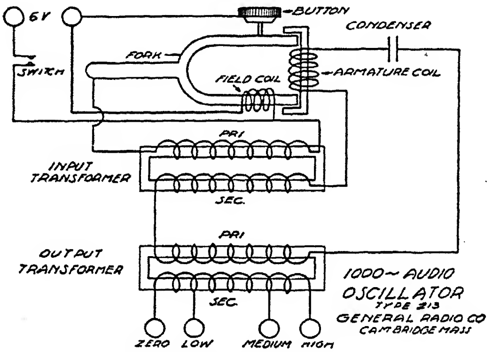

FIG. 88.—Wiring diagram for audio-oscillator.

the primary of the input transformer induces an E.M.F. in the secondary which starts oscillations in the resonating circuit which includes, besides the condenser and primary of the output transformer, the armature coil. This oscillating current changes the attraction between the armature pole tips and the prongs of the fork. Since the secondary circuit is tuned to the period of the fork, the fork resonates to it, thus building up a vigorous vibration. The microphone button, being in contact with the fork, supplies a varying current of this same frequency to the primary of the input transformer and energy from the battery is thus furnished to maintain the oscillations, and carry the load put upon the secondary of the output transformer.

Each transformer coil has a small air gap to prevent distortion, but their magnetic circuits are sufficiently closed to prevent disturbing stray fields. The oscillator is self starting and may be placed at some distance and operated by a key near the bridge. The coils are so wound that a 6-volt battery furnishes ample

power. The device is not designed to furnish more power than that required by a single bridge circuit. If overloaded, the microphone is likely to pack. It is carried by a stiff spring mounted on one prong of the fork and its inertia is sufficient to insure response to vibrations of the fork.

120. The Vreeland Oscillator.—None of the sources thus far described produce alternating E.M.F.'s of a purely sinusoidal wave form. There are a number of important bridges which do

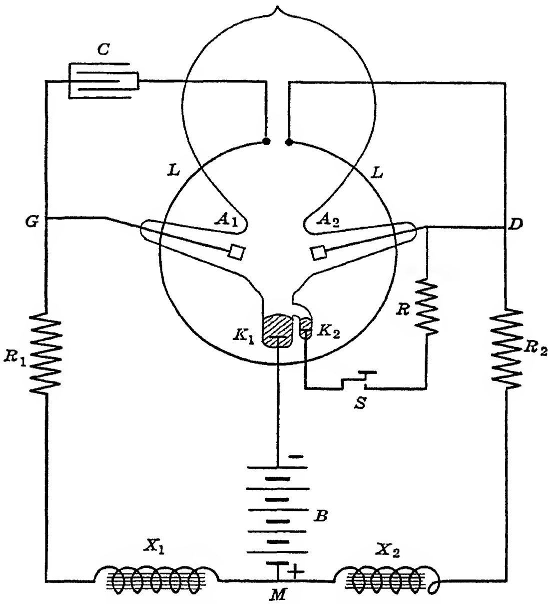

Fig. 89.—Wiring diagram for Vreeland oscillator.

not balance independent of the frequency and when a telephone is used as the detecting device, complete silence can not be obtained with impure wave forms. In such cases, when the fundamental has been balanced out, the overtones are still heard and materially mar the sharpness of setting which would otherwise be possible.

The Vreeland Oscillator is one of the best sources of pure sine waves available. It is, in reality, a mercury arc rectifier operated backwards, the connections for which are shown diagrammatically in Fig. 89. The essential part of the device is a large pear shaped mercury arc tube with two anodes A_1 and A_2 having a common

mercury cathode K_1. It is well known that the mercury arc will operate only when the mercury electrode is negative. When used as a rectifier, the condenser and deflecting coil are removed and the source of alternating E.M.F. is connected to the terminals $G.D.$ When G is positive and D negative, current will flow from A_1 to K_1 through the battery B, which is here shown as the load, to D, and when D is positive, the path is $A_2K_1\ MG$, these furnishing a current through B in the same direction as before. The reactances X_1 and X_2 serve to smooth out the fluctuations through the battery.

To understand its operation as an oscillator, let us suppose that the source of $A.C.$ is removed and that the deflecting coil and condenser are connected to G and D as shown in the figure. The battery B now becomes the source of power. An arc is started between the electrodes K_1 and K_2 by shaking the tube slightly, thus causing the mercury pools to unite and break again. The tube is quickly filled with ionized mercury vapor and the arc spreads to the anodes A_1 and A_2. The switch S is then opened thus stopping the arc to K_2. If the impedance of the two paths MD and MG are equal and the tube is symmetrical, the arc will divide equally between the anodes A_1 and A_2 which are thus at the same potential and there is no charge in the condenser. If, however, some irregularity in the tube causes more current to flow momentarily to the anode A_1 it will be at a higher potential than A_2 and a charging current will flow to the condenser through the deflecting coil LL. This coil, which really consists of two parts, one in front of the tube and the other behind it, is placed so that its magnetic field is perpendicular to the flow through the arc. If the polarity is so chosen that the charging current deflects the arc stream so as to further increase the current to A_1 a very appreciable charge may be given to the condenser. When the condenser discharges, the deflecting action of the current which is now reversed will cause more current to flow to the anode A_2 thus raising its potential above A_1 and charging the condenser in the opposite direction. The deflecting coil serves the double purpose of furnishing a self inductance to form, with C, an oscillatory circuit, and to automatically deflect the arc streams from one anode to the other to maintain the oscillations. The frequency is given by the expression

$$n = \frac{1}{\ }$$

where L is the inductance of the deflecting coil in henries, and C the capacitance of the condenser in farads. It is found that such a device, when properly designed, will oscillate at frequencies ranging from 100 to 4,000 cycles per second.

Another coil, placed near the deflecting coils, serves as the secondary of an air cored transformer to supply current to a bridge. It is found that the frequency is but little affected by changes in the load on the secondary. Because of the relatively large coils, the instrument possesses an appreciable stray field and must be placed at some distance from the bridge, to prevent direct induction in the coils which are being studied.

121. The Electron Tube Oscillator.—One of the simplest and most effective means of obtaining alternating voltage of any desired frequency is that in which a three element electron tube is used to maintain continuous oscillations in a resonance circuit. The underlying principle is the amplifying action of the tube which will be described in Chap. XVI. It will be sufficient for the present purpose to point out that the electron tube consists of a highly evacuated glass container in which are placed a filament and a metal plate with a grid mounted between them. The grid consists of a fairly coarse meshed structure of fine wires. When the filament is heated to incandescence by an electric current, it emits electrons which may be drawn to the plate by a battery connected through an external circuit between the plate and filament. The positive terminal of the battery must be connected to the plate.

Inasmuch as the electrons, to reach the plate, must pass through the meshes of the grid, the number arriving at the plate may be controlled by giving suitable potentials to the grid, and may be stopped entirely, if the grid is sufficiently negative. Inasmuch as the energy required to maintain a given potential on the grid is small, the device acts as an electrical throttle valve, whereby the available energy of the plate circuit battery may readily be controlled. Figure 120 of Chap. XVI shows the relation which exists between the plate current and grid volts. The time required for an electron to traverse the distance from filament to plate depends upon the potential of the plate but is of the order of 10^{-8} seconds. Changes in plate current accordingly follow changes in grid volts with remarkable swiftness.

There are many different circuits in which an electron tube may be used to generate sustained oscillations. Figure 90 shows one

of the simplest. F, G, and P are the filament, grid and plate respectively. The filament is heated by the battery A, whose current is controlled by the rheostat R. The battery B furnishes the potential to draw the electrons from the filament to the plate. The inductance L and the condenser C form the oscillatory circuit. To understand the way in which oscillations are sustained, let us suppose that, by closing the switch S, the establishment of a current through the coil L has produced a transient oscillation in the circuit LC. This would quickly die out if energy were not supplied to it to compensate for the losses. Suppose that the oscillatory current through L is in the direction of the arrow and is rising. Due to the self inductance L there will be an E.M.F. in the coil in the direction DE. This lowers the potential of the grid with respect to the filament which thus decreases the plate current, flowing through the part L_p. This decrease in the plate current induces in L_p an E.M.F. which tends to keep the oscillatory current flowing. A continuation of this reasoning throughout the changes occurring during a complete cycle will show that the variations in plate current always induce in L_p an E.M.F. tending to drive the oscillatory current in L in the direction in which it happens to be flowing at any instant. The oscillations would increase indefinitely in amplitude were it not for the fact that the grid volt-plate current characteristic of the tube becomes horizontal at each end.

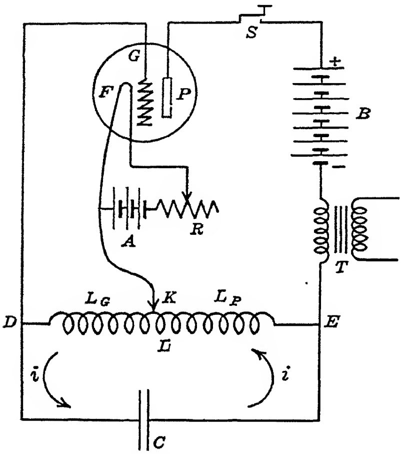

FIG. 90.—Electron tube oscillator.

Frequencies ranging from 1 cycle to several millions per second may be obtained. Alternating current power for bridge work may be obtained either by placing another coil near L which then serves as the secondary of an air core transformer or by connecting the primary of a telephone transformer in the plate circuit as shown in the figure. The latter is to be preferred since variations in the load have a smaller disturbing effect upon

the frequency than is the case with the former arrangement. The wave form is not as free from harmonics as that obtained from a Vreeland oscillator, and a filter must be used in cases where extreme purity is essential.

DETECTING DEVICES

122. Telephone Receiver.[1]—The telephone receiver is one of the most generally useful of the various instruments for detecting the balance condition in a bridge circuit actuated by alternating currents. It consists essentially of a horseshoe magnet upon which is wound a pair of coils carrying the current to be detected, and a soft iron diaphragm mounted near the poles as shown in Fig. 91. The current through the coils magnetizes the core which attracts the diaphragm with a force proportional to the square of the induction produced. The sensitivity of the receiver is increased by using for the core, not a piece of soft iron, but a permanent magnet. The way in which this is brought about may be seen from the following consideration. Let B_0 be the constant induction through the gap due to the permanent magnet, and let the additional induction which is proportional to the current i in the coils be $k_1 i$. Then the total pull on the diaphragm is given by

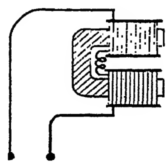

Fig. 91.—Telephone receiver.

$$\text{Pull} = k_2 B^2 = k_2(B_0 + k_1 i)^2 = k_2 B_0^2 + 2k_1 k_2 B_0 i + k_1^2 k_2 i^2$$

The first term represents the pull due to the permanent magnet alone, the second, that due to the current and magnet combined, while the third is that due to the current alone. If it is desired to have the motion of the diaphragm follow the variations in the current so that its motions may reproduce for the human ear the sound waves acting upon the diaphragm of a distant telephone transmitter, then the receiver must be so designed that the second term is large compared to the last which contains the square of the current. The first term need not be considered since it is independent of the current. The desired effect is attained by making B_0 large compared to $k_1 i$. Since B_0 enters as a factor in the second term, making it large has the effect of

[1] Mills, Radio Communication, p. 27.

increasing the motion of the diaphragm and hence of making the receiver, to a certain extent, an amplifying device.

If the third term is not negligible compared to the second, then, although there is a repetition with amplification there is also distortion since the pull which it defines is proportional to the square of the current. The nature of this distortion can be understood by supposing that the current is sinusoidal, e.g., $i = I \sin \omega t$. The last term then becomes

$$k_2k_1^2I^2 \sin^2 \omega t = k_2k_1^2I^2 \; {}^{1 - \cos 2\omega t}$$

and it is seen that the distorting pull is made up of two parts: A constant part $\frac{k_2k_1^2I^2}{2}$ which need not be considered and a pulsating part having twice the frequency of the phone current.

Since the diaphragm of the telephone receiver is an elastic body it will have a frequency of its own and will accordingly respond more vigorously to frequencies which correspond to its natural period, and another source of distortion is thus introduced. For bridge work, however, this fact may be utilized to increase the sensitivity by impressing upon the bridge the frequency to which the telephone resonates. Phones for this particular purpose are constructed in such a way that their resonance frequencies may be varied over a considerable range.

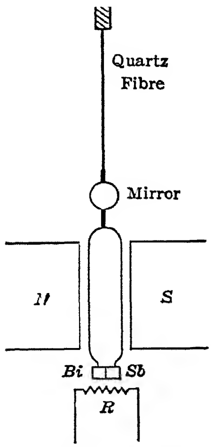

FIG. 92.—Duddell thermo-galvanometer.

123. Thermo-galvanometer.—The Duddell Thermo-galvanometer is an adaptation of the Boys' radio-micrometer for the purpose of measuring and detecting small alternating currents. The moving system, shown in Fig. 92, consists of a single turn of silver wire at the bottom of which is a tiny thermocouple of bismuth and antimony. The system is suspended by means of a fine quartz fibre between the poles of a strong horseshoe magnet and carries a small mirror by means of which its deflections are read with a lamp and scale. Immediately below the thermo-junction is mounted a resistance unit through which the current to be measured is passed. The heat from this current is carried to the thermo-junction by convection

and radiation and causes a current to flow through the low resistance silver loop which is deflected by the electrodynamic action of the field. Since the heating effect is proportional to the square of the current while the thermal E.M.F., for small temperature differences, is proportional to the temperature, the indications of this instrument are roughly proportional to the square of the current.

Several heating units are provided with each instrument and range in value from 1 to 1,000 ohms according to the current sensitivity desired. For low resistances, they are made of fine wire bent back and forth but for the higher values, fine platinized quartz fibres are used. With the latter, current sensitivities of 10^{-5} amperes are obtained. This type of instrument may be calibrated on direct currents and then used to measure alternating currents. Since it is practically free from inductance, the instrument may be used for the measurement of currents of very high frequencies. Because of the low resistance of the moving system it is critically damped electromagnetically, and is usually designed so as to have a period of from three to four seconds. Because of the delicacy of the quartz fibre suspension and the light silver loop, it is not a robust instrument and must be handled with caution. The heating elements are easily burned out and should always be protected by a high resistance which may be reduced to zero when it has been ascertained that safe limits of current will not be exceeded. Sudden changes in temperature cause the zero to drift and the instrument is usually enclosed in a tight wooden case.

124. Vibration Galvanometer.[1]—The vibration galvanometer is one of the most useful instruments available for the detection of minute alternating currents of commercial frequencies. To secure suitable sensitivity, advantage is taken of the principle of resonance. That is, the moving system is so adjusted mechanically, that its natural period coincides with that of the alternating current to be detected. Although the instrument shows very little response to direct currents or to alternating currents to which it is not tuned, nevertheless when resonance has been secured, a very appreciable vibration results. The vibrations are indicated by means of a small concave mirror carried on the moving system which focuses the image of an incandescent fila-

[1] LAWS, Electrical Measurements, p. 434.
WENNER, Bull. U. S. Bureau of Standards, vol. 6, 1909–10, p. 347.

ment on a ground glass scale. When the system vibrates, the image is drawn out into a broad band of light, while very slight motions are detected by a diminution in the sharpness of the line.

One of the chief reasons for the superiority of this instrument is the fact that its response is selective. In many measurements it is necessary to use a pure sine wave, a thing difficult to secure. Since vibration galvanometers may be made with a selectivity

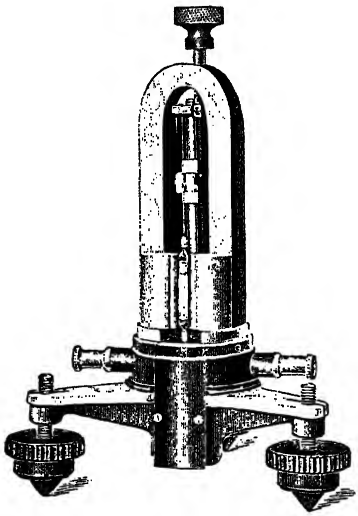

Fig. 93.—Leeds and Northrup vibration galvanometer.

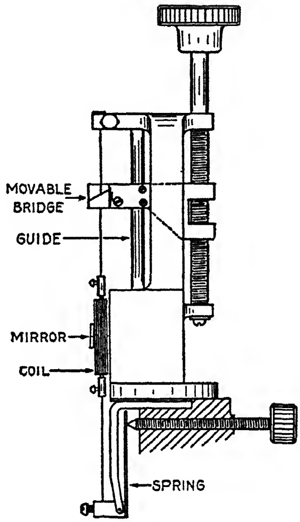

Fig. 94.—Tuning mechanism for Leeds and Northrup vibration galvanometer.

so high that their response to the third harmonic is $\frac{1}{4,000}$ of that to the fundamental and to the fifth, $\frac{1}{12,000}$, impure waves may be employed with very little if any inaccuracy introduced. In fact an interrupter of the vibrating wire type described above, giving a square wave form, may be employed. The current sensitivity of the vibration galvanometer is about the same as that of a good telephone receiver—i.e., 10^{-6} amperes.

Obviously the instrument may be of either the D'Arsonval or the Thomson type. In Fig. 93 is shown one of the former, or moving coil instruments, while Fig. 94 shows how the moving

system is tuned. The coil is held in position by a taut phosphor-bronze ribbon, the effective length of which is varied by means of the movable bridge carried on the upper screw. By sliding this bridge up or down rough tuning is obtained while fine adjustments are secured by slightly changing the tension of the suspension by means of the lower screw and spring.

Figure 95 shows a Tinsley instrument of the moving magnet type. The vibrating system consists of a small permanent magnet mounted on a taut metallic ribbon behind which is held

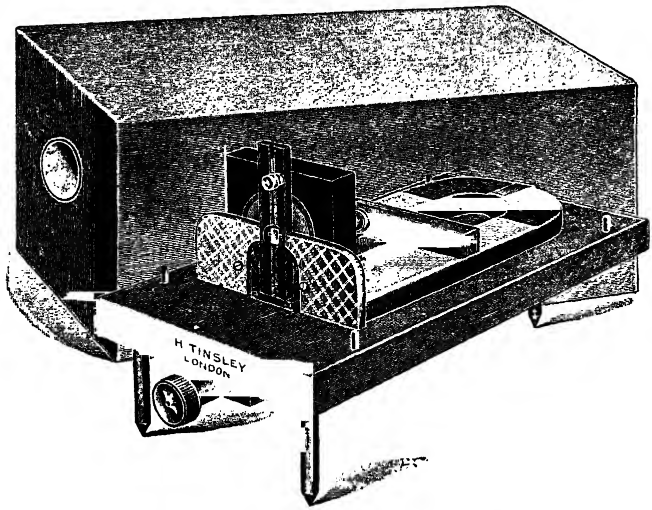

FIG. 95.—Tinsley vibration galvanometer.

the fixed deflecting coil. Specially shaped pole pieces concentrate the field of the large horseshoe magnet on the moving magnet. Since the period of the system is determined largely by the strength of this external field, tuning is obtained by changing this field. This is accomplished by moving the soft iron magnetic shunt along the horseshoe magnet. The milled head shown at the front of the base operates a worm gear which moves the shunt.

125. Alternating Current Galvanometer.—The alternating current galvanometer is one of the most sensitive devices available for detecting the balance condition in a bridge supplied with an alternating E.M.F. It is essentially a D'Arsonval galvanometer with the permanent magnet replaced by an electro-

magnet energized from the same A.C. source as that supplying the bridge. It operates upon the electrodynamometer principle and the direction of the torque acting upon the moving coil is independent of the polarity of the supply. Its operation is complicated by the fact that, when connected to the bridge, there are present in the coil two currents, one due to the unbalanced condition of the bridge, and one induced by the alternating flux of the galvanometer field. Since the former is small and disappears at balance, the latter by far overpowers it and must either be eliminated or made ineffective.

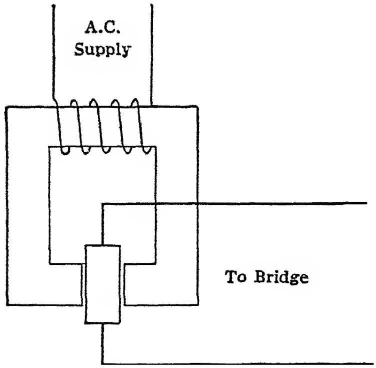

FIG. 96.—Alternating current galvanometer.

It may be shown in the following manner that when the current induced in the coil is 90° out of phase with the flux through it, the torque is zero.

Let $\phi = \Phi \sin \omega t$ = instantaneous flux through the coil and $i = I \sin(\omega t \pm \theta)$ instantaneous current in the coil.

The torque acting upon the coil in a given position at any instant is then

$$\tau = K\Phi \sin \omega t \, I \sin (\omega t \pm \theta)$$

where K is a constant depending upon the geometry of the instrument. The average value $\bar{\tau}$ taken over the time T of one complete cycle is

$$\bar{\tau} = \frac{K\Phi I}{T}\int_o^T \sin \omega t \sin (\omega t \pm \theta) dt$$

$$= \frac{K\Phi I}{T}\int_o^T \sin \omega t(\sin \omega t \cos \theta \pm \cos \omega t \sin \theta) dt$$

$$= \frac{K\Phi I}{T} \qquad \theta \int^T \sin^2 \omega t dt \pm \sin \theta \int_o^T \sin \omega t \cos \omega t dt \Big]$$

$$K\Phi I \qquad \theta \int_o^T \frac{1 - \cos 2\omega t}{2} dt \pm \frac{\sin \theta \sin^2 \omega t}{2}\Big]_o^T = \qquad \cos \theta$$

The resultant torque on the coil is, accordingly, positive or

negative depending upon the sign of θ and is zero for $\theta = \pm\frac{\pi}{2}$. Since the E.M.F. induced in the coil is 90° out of phase with the flux producing it, the condition stated above is equivalent to saying that the induced current must be in phase with the E.M.F. In other words, the bridge circuit to which the coil is connected must be nonreactive. In certain bridges such as those for comparing two condensers or two inductances this is obviously impossible. The required condition may, however, be met by shunting across the coil an appropriate variable reactance, e.g., an inductance with a variable series resistance in the former case, or a condenser and resistance in the latter.

When the galvanometer is first connected to the bridge, it will be found that, due to the action just described, the coil assumes a very rigid position, including either a maximum or a minimum amount of the field flux. If the former position results, a leading current through the coil is indicated, and a shunt with an inductive reactance must be applied. For satisfactory operation, a certain amount of stability is required to give a constant zero position, so inductive reactance across the coil should predominate. Since the reactance of the bridge is an important factor in determining the rest position of the coil, the galvanometer key must remain closed, and the balance established by opening and closing the supply circuit to the bridge.

The alternating current galvanometer has an important advantage over detecting devices such as the telephone or vibration galvanometer, in that it swings to the right or left according to the phase of the current at the galvanometer corners of the bridge while with the latter, no such effect is possible. Furthermore, if a direct current is supplied to the field, it becomes an ordinary D'Arsonval galvanometer and may be used to determine the steady state balance. Its sensitivity may be made 100 times that of the telephone or vibration galvanometer. It has, however, one distinct disadvantage, in that the deflection depends not only upon the field and current through the coil but also upon the phase angle between them. It can not therefore be calibrated to measure currents. Furthermore, zero deflection indicates either no current, or current 90° out of phase with the field. A simple test for the latter condition is to shift the phase of the field by inserting a resistance in series with the field coil.

CHAPTER XII

ALTERNATING CURRENT BRIDGES

126. General Considerations.—In order to obtain the reactive effect of an inductance or a capacitance it is necessary that the current through it should be variable. In the early bridge measurements for comparing inductances or capacitances and even for determining an inductance in terms of a capacitance the variable current was obtained simply by closing and opening the battery circuit leaving the galvanometer permanently connected to the bridge. The galvanometer employed was usually of the long period ballistic type. This procedure is open to two objections. First, the sensitivity thus realizable is not great and second it may lead to results which are appreciably different from the effective values of the condensers or coils when employed, as is usually the case, in circuits traversed by alternating currents. For example, the effective value of the self inductance of the primary of a transformer when an alternating current is flowing through it, depends upon the load across the secondary. If measured by the make and break method with a ballistic galvanometer as the detecting device, the result is the inductance of the primary alone independent of the effect of the secondary.

The ballistic galvanometer is an integrating instrument, and a zero deflection does not necessarily mean that no current has passed through it, but that equal and opposite quantities have traversed it. The bridge may have been out of balance each way during the time the current through it was changing. It is, accordingly, much better to use alternating currents through the bridge and employ a detecting device such as the telephone or vibration galvanometer, a zero indication of which indicates that at no time is there a current through it, and that the bridge is balanced at all times.

It will appear in the discussion which follows that, in order for a

bridge with reactive members to be balanced at all times, there are two conditions which must be satisfied. First, the bridge must be balanced for direct currents, "steady state balance," and second, it must be balanced for alternating currents, "variable state balance." These two balance conditions may be interpreted in the equation for the bridge in a simple manner. An expression is deduced, involving one current and its time derivative. The "steady state balance" means that the coefficient of the current term is zero. The "variable state balance" means that the coefficient of the term for the changing current, i.e., the time derivative, is zero. This applies to any bridge for which the balance condition may be reduced to an expression involving only one current and its first time derivative. Such a bridge balances independent of the frequency. If a second time derivative is involved, as, for example, in Trowbridge's bridge and the frequency bridge, the wave form of the current must be assumed, and the bridge no longer balances independent of the frequency. In some instances, the student will find it advantageous to obtain the former by the use of a battery and direct current galvanometer, and later apply an alternator and detector for the variable state balance. After becoming experienced in this type of work, however, both balances may be obtained simultaneously by the use of alternating currents.

127. Maxwell's Bridge.[1]—One of the simplest methods for determining an inductance in terms of a capacitance or vice versa is the method known as Maxwell's bridge. It consists of an ordinary Wheatstone's bridge with three non-inductive resistances R_1, R_2, and R_3, as shown in Fig. 97, while the fourth arm contains the inductance L to be determined. Let the ohmic resistance of this coil be R_4. To offset the reactance of the coil L, a condenser C is placed across the opposite resistance R_1. When an alternating E.M.F. is applied to the bridge, the current in the upper half will lead the E.M.F., while that in the lower half lags behind it. Accordingly, if an A.C. galvanometer or other detecting device is connected ahead of the inductance and behind the condenser, the arms of the bridge may be so adjusted that the potential changes at D and E are

[1] MAXWELL. Electricity and Magnetism, vol. 2, p. 387.

not only equal but also in phase, and no indication of the instrument will result.

The conditions necessary for balance may be obtained in the following manner. Let the instantaneous currents through the various elements be designated as in the figure. By equating

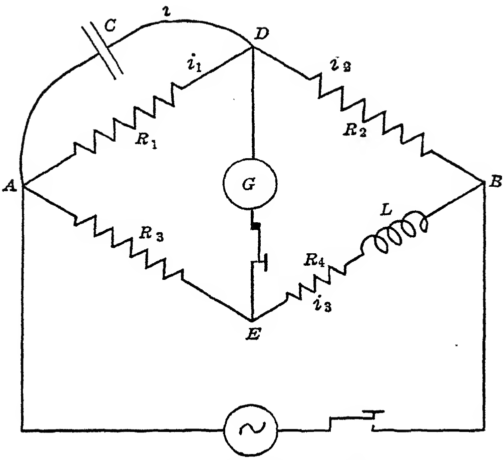

Fig. 97.—Maxwell's bridge.

the fall of potential from A to D to that from A to E, and the fall from D to B to that from E to B and noting that i_2 is made up of i and i_1 the following equations result.

$$i_2 = i_1 + i \tag{1}$$

$$R_1 i_1 = R_3 i_3 \tag{2}$$

$$R_2 i_2 = R_4 i_3 + L\frac{di_3}{dt} \tag{3}$$

$$R_1 i_1 = \frac{1}{C}\int i\,dt \tag{4}$$

We thus obtain four equations between the four currents. The currents may therefore be eliminated and the relations between L, C, and the R's obtained which are necessary for a balance. Eliminating i_2 between eqs. (1) and (3) there results

$$R_4 i_3 + L\frac{di_3}{dt} = R_2(i_1 + i) \tag{5}$$

Differentiating eq. (4) with respect to t and solving for i, also substituting the value of i_3 from eq. (2) in eq. (5), we have

$$\frac{R_4R_1}{R_3}i_1 + L\frac{R_1di_1}{R_3\,dt} = R_2\left((i_1 + R_1C\frac{di_1}{dt}\right) \tag{6}$$

If the bridge has first been balanced for the steady state, $\frac{R_4R_1}{R_3} = R_2$, whence only the terms containing the derivative of i_1 remain. The second condition for balance is obtained by equating the coefficients of the derivatives, whence

$$L = R_2R_3C \tag{7}$$

While the theory of this bridge is simple, its application in the laboratory is somewhat tedious in case both L and C are fixed. For example, suppose a steady state balance has been obtained, and it is attempted to satisfy eq. (7) by changing R_2 or R_3. The steady state balance is immediately upset and must again be obtained before the test for the new value of R_2 or R_3 can be made. If L or C are continuously variable, eq. (7) may be satisfied without disturbing the steady state balance, and it is in this case that the bridge is particularly useful. An experienced observer however, quickly learns to make both balances simultaneously.

128. Experiment 20. *Maxwell's Bridge for Self Inductance.*—Make the connections as shown in Fig. 97 using for L a continuously variable inductance and for C a subdivided condenser. As a source of E.M.F., use an alternator giving a frequency from 500 to 1,000 cycles per second, and a pair of head phones as the detector. It may be well to use a battery and ordinary galvanometer to obtain the steady state balance. Connect double pole double throw switches so that each source and detector may be quickly exchanged. For the steady state balance, care must be taken to close the battery key before the galvanometer key. Obtain the variable state balance by changing L. If the balance does not lie within the range of L, change either C or one of the resistances of eq. (7). If the latter is done, a new steady state balance must be obtained.

Report.—Plot a calibration curve of L as a function of its scale readings. Define coefficient of self inductance. If a copper disk were held near the coil so that its face is perpendicular to the axis of the coil would the inductance as measured in this manner be changed? Explain.

129. Anderson's Modification of Maxwell's Bridge.[1]—It was pointed out above that the adjustments for balancing Maxwell's bridge are likely to be tedious because each attempt to obtain a variable state balance necessitates a redetermination of the steady state balance. Anderson has suggested a simple device by which the variable state balance may be obtained without destroying that for steady states. The connections are shown in Fig. 98. It will be noted that the condenser C, instead of being connected to the point D has the resistance r, placed in

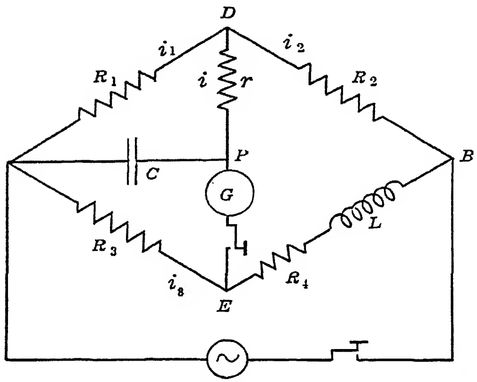

FIG. 98.—Anderson's modification of Maxwell's bridge.

series with it so that its time constant may be varied. Since, in determining the steady state balance, the condenser produces no effect, it may be left in circuit during that process, and the only change introduced is placing r in series with the galvanometer. This reduces slightly the sharpness of balance which is of little consequence. The steady state balance may accordingly be made once for all, and the variable state balance obtained by adjusting r to the proper value.

The determination of the balance condition is somewhat more complicated and is as follows: Let the instantaneous currents through the various elements of the bridge be designated as before. The points P and E are now the ones remaining at the same potential. Accordingly,

$$i_2 = i_1 + i \tag{1}$$

$$R_1 i_1 = \frac{1}{C}\int idt + ri \tag{2}$$

[1] *Phil. Mag.*, vol. 31, 1891, p. 329.

$$\frac{1}{C}\int idt = R_3 i_3 \tag{3}$$

$$R_2 i_2 + ri = L\frac{di_3}{dt} + R_4 i_3 \tag{4}$$

Combining eqs. (1) and (3) with eq. (4), there results

$$R_2(i_1 + i) + ri = \frac{L}{R_3 C}\, i + \frac{R_4}{R_3 C}\int idt \tag{5}$$

Eliminating i_1 between eqs. (2) and (5) we have

$$R_2\left[\frac{1}{R_1 C}\int idt + \frac{r}{R_1} i + i\right] + ri = \frac{L}{R_3 C}\, i + \frac{R_4}{R_3 C}\int idt \tag{6}$$

Imposing now the condition for the steady state balance, it is seen that the coefficients of the integrals are equal and eq. (6) then becomes

$$\frac{R_2 r}{R_1} + R_2 + r = \frac{L}{R_3 C} \tag{7}$$

Rearranging and using

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}$$

$$L = C\Big[(R_3 + R_4)\, r + R_2\, R_3\Big] \tag{8}$$

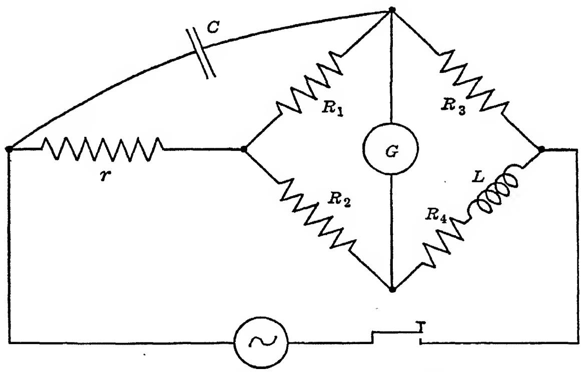

FIG. 99.—Stroude and Oates bridge.

Another change in the arrangement of this bridge has been suggested by Stroude and Oates[1] which is usually an advantage. The general theory of bridges shows that it is always possible to interchange the source of power and the detecting device. Figure 99 shows the connections when this has been done with a slight

[1] *Phil. Mag.*, vol. 6, 1903, p. 707.

change in the arrangement which improves the ease of manipulation. The principal advantage in this method lies in the fact that r is now in series with the bridge and a correspondingly higher E.M.F. may be used without injuring the resistances. An increase in sensitivity is thus secured. The same balance condition, eq. (8), applies.

130. Experiment 21. *Stroude and Oates Bridge for Self Inductance.*—Connect the apparatus as shown in Fig. 99. For power supply and detector use either an audio frequency generator and phones or city A.C. supply and alternating current galvanometer. The latter is particularly well adapted to this bridge. Arrange double pole double throw switches so that a direct current source and ordinary galvanometer may quickly be substituted for making the steady state balance.

As an unknown inductance, use two coils mounted in a fixed position close enough to one another so that mutual inductance exists between them. Measure the inductance of each separately, then connect them in series and measure the resultant inductance with the connections direct and reversed, that is with the mutual inductance first aiding and then opposing the self inductances. Calling L_1 and L_2 the self inductances of the individual coils, and L_a and L_o the two together when aiding and opposing respectively, the following equations hold

$$\begin{aligned} L_a &= L_1 + L_2 + 2M \\ L_o &= L_1 + L_2 - 2M \end{aligned} \tag{9}$$

Report.—Check your results by solving eq. (9) for M. Give a physical interpretation for eq. 9.

131. Trowbridge's Method for Self Inductance.—In Art. 141 there will be described a method by which an inductance may be measured in terms of capacitance using the two reactances in series in one arm of a bridge. While this arrangement admits of an exceedingly sharp adjustment, the bridge may be balanced for only one frequency for given values of L and C. In fact one of its most useful applications is the determination of frequency using reactances of known magnitudes. Trowbridge[1] has shown that if the reactances are shunted with properly chosen resistances the balance condition may be made independent of the frequency while sensitiveness of balance is very inappreciably sacrificed. Such an arrangement is shown in Fig. 100.

[1] *Phys. Rev.*, vol. 23, 1905, p. 475.

Let the currents be designated as indicated in the figure. Then putting, for the moment, $R_4 = 0$, the following equations must hold for the balance condition:

$$R_1 i_1 = R_2 i_2 \tag{1}$$
$$R_3 i_1 = R i_4 + r i_6 \tag{2}$$
$$R i_4 = r_o i_3 + L\frac{di_3}{dt} \tag{3}$$
$$r i_6 = \frac{1}{C}\int i_5 dt \tag{4}$$
$$i_2 = i_3 + i_4 \tag{5}$$
$$i_2 = i_5 + i_6 \tag{6}$$

For simplicity let $R_1 = R_2$, then $i_1 = i_2$.

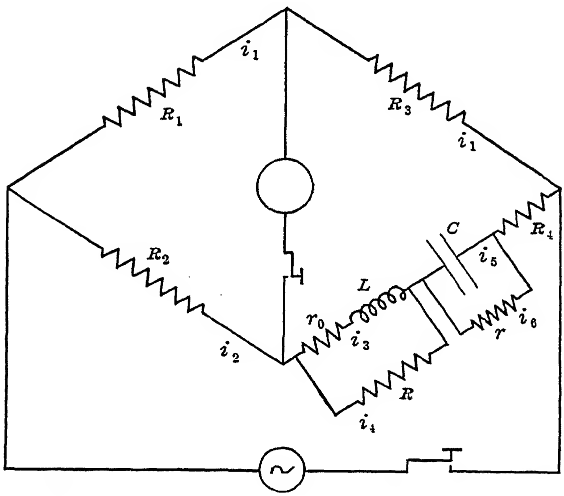

FIG. 100.—Trowbridge's method for self inductance.

Eliminating i_6 between eqs. (2) and (6) we have

$$(R_3 - r)\, i_1 = R i_4 - r i_5 \tag{7}$$

Again eliminating i_6 between eqs. (2) and (4) and substituting the value of i_5 in eq. (7)

$$(R_3 - r) i_1 = R i_4 - r\left[R_3 C\frac{di_1}{dt} - RC\frac{di_4}{dt}\right] \tag{8}$$

Substituting the values of i_1 and i_4 in terms of i_3 obtained from eqs. (3) and (5), namely,

$$i_4 = \frac{r_0}{R} i_3 + \frac{L}{R} \frac{di_3}{dt}$$

$$i_1 = i_3 + i_4 = \left(\frac{R + r_0}{R}\right) i_3 + \frac{L}{R} \frac{di_3}{dt}$$

in eq. 8 there results

$$(R_3 - r)\left[\frac{R + r_0}{R} i_3 + \frac{L}{R} \frac{di_3}{dt}\right] = r_0 i_3 + L \frac{di_3}{dt} - rR_3C\left[\frac{R + r_0}{R} \frac{di_3}{dt} + \frac{L}{R} \frac{d^2 i_3}{dt^2}\right] + rRC\left[\frac{r_0}{R} \frac{di_3}{dt} + \frac{L}{R} \frac{d^2 i_3}{dt^2}\right] \quad (9)$$

Collecting terms we have,

$$\left[(R_3 - r)\frac{R + r_0}{R} - r_0\right] i_3 + \left[\frac{(R_3 - r)L}{R} - L + rR_3C \frac{R + r_0}{R} - rr_0C\right] \frac{di_3}{dt} + \left[(R_3 - R)\frac{rCL}{R}\right] \frac{d^2 i_3}{dt^2} = 0 \quad (10)$$

Clearing of fractions,

$$[(R_3 - r)(R + r_0) - Rr_0)] i_3 + [(R_3 - r - R)L + rR_3(R + r_0)C - Rrr_0C] \frac{di_3}{dt} + (R_3 - R)rLC \frac{d^2 i_3}{dt^2} = 0 \quad (11)$$

Assuming now that i_3 is an alternating current of the form

$$i_3 = I \sin \omega t$$

and substituting this in eq. (11) there results

$$[(R_3 - r)(R + r_0) - Rr_0] I \sin \omega t + [(R_3 - r - R)L + rR_3 (R + r_0)C - Rrr_0C] I\omega \cos \omega t - (R_3 - R) rLCI\omega^2 \sin \omega t = 0 \quad (12)$$

In an ordinary measurement in which C is expressed in microfarads and L in millihenries the last term is of the order 10^{-9} and may be neglected without appreciable error. Since eq. (12) holds for all values of t, we have

when $t = 0$,

$$(R_3 - r - R) L + [rR_3(R + r_0) - Rrr_0] C = 0$$

whence

$$L = \frac{[Rrr_0 - (R + r_0)rR_3]C}{R_3 - R - r} = \frac{[rr_0(R - R_3) - RR_3r]C}{R_3 - R - r} \quad (13)$$

when

$$t = \frac{\pi}{2}, \quad (R_3 - r)(R + r_0) - Rr_0 = 0 \quad (14)$$

which is seen to be the condition for a steady state balance. If $R_3 = R$, the last term of eq. (12) vanishes, and the expression given by eq. (13) is exact and reduces to

$$L = \frac{RR_3rC}{r} = RR_3C = R^2C \tag{15}$$

The bridge, when used in this manner, is well adapted to the standardization of a variable inductance such as Brooks inductometer but is not well suited to cases in which the values of L and C are fixed or variable by steps, since it is impossible to adjust for the variable state balance without upsetting that for steady states. The author has pointed out that this difficulty may be avoided by use of the resistance R_4 as shown in the figure. If identical boxes are employed for r and R_4 and a steady state balance obtained with R_4 set at a suitable value, then the variable state balance may be obtained by shifting plugs from one box to the other, keeping $r + R_4$ constant. If an equal arm bridge is used, this has the effect merely of adding to both the upper and lower right hand arms of the bridge the value of R_4. Eq. (13) then becomes

$$L = \frac{[rr_0\,(R - R_3 + R_4) - Rr(R_3 - R_4)]C}{R_3 - R_4 - R - r} \tag{16}$$

132. Experiment 22.—*Trowbridge's Method for Self Inductance.* Connect the apparatus as shown in Fig. 100 using the telephone and suitable oscillator for detector and energy source respectively. As an unknown use a smoothly variable inductance, and set the four resistances R_1, R_2, R_3, and R at suitable values, e.g., 500 ohms each. C should be a subdivided standard condenser. Measure the unknown for several settings and plot its calibration curve. Replace the variable inductance by one of fixed value, and measure it, making use of the resistance R_4 as explained above.

133. Heydweiller's[1] Network for Mutual Inductance.—In Exp. 18, a method due to Carey-Foster, was used for the measurement of mutual inductance in terms of capacitance. The essential feature of this method consists in balancing the charge of a condenser against the quantity of electricity induced in the secondary of a mutual inductance when a certain current change takes place in the primary. This balance was effected by discharging the two quantities involved in opposite directions through a long period ballistic galvanometer. While this circuit is satisfactory

[1] *Annalen der Physik.*, vol. 53, 1894, p. 499.

when used with the make and break method of excitation, it can not be used with alternating currents since there is no way of adjusting the time constant of the condenser circuit.

This defect was overcome by Heydweiller by the introduction of the resistance S as shown in Fig. 101 and a very satisfactory method for the measurement of mutual inductance was thus obtained. The resistance P includes that of the secondary coil whose self inductance is L. The conditions which must hold

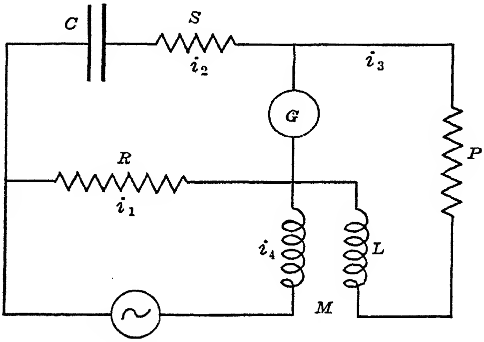

FIG. 101.—Heydweiller's network for mutual inductance.

for zero current through the galvanometer may be obtained as follows. Designating the instantaneous currents through the various resistances as indicated in the figure, we have

$$i_2 = i_3 \tag{1}$$

$$i_4 = i_3 + i_1 \tag{2}$$

$$Ri_1 = \frac{1}{C}\int i_2 dt + Si_2 \tag{3}$$

$$L\frac{di_3}{dt} + Pi_3 = M\frac{di_4}{dt} \tag{4}$$

From eqs. (2) and (4) we have

$$L\frac{di_3}{dt} + Pi_3 = M\left(\frac{di_3}{dt} + \frac{di_1}{dt}\right) \tag{5}$$

From eqs. (3) and (1)

$$i_1 = \frac{1}{RC}\int i_3 dt + \frac{S}{R}i_3 \tag{6}$$

Differentiating eq. (6) and substituting in eq. (5), there results, on collecting terms,

$$\left[L - M\frac{(R + S)}{R}\right]\frac{di_3}{dt} + \left[P - \frac{M}{RC}\right]i_3 = 0 \tag{7}$$

Since an alternating E.M.F. is applied to this circuit the current i_3 is also alternating and may be represented by

$$i_3 = I \sin \omega t; \text{ whence } \frac{di_3}{dt} = I\omega \cos \omega t \qquad (8)$$

Substituting these values in eq. (7) we have

$$\left[L - M\frac{(R + S)}{R}\right] I\omega \cos \omega t + \left[P - \frac{M}{RC}\right] I \sin \omega t = 0 \quad (9)$$

Since eq. (9) holds for all values of t, we have

when

$$\omega t = 0, L - M\frac{(R + S)}{R} = 0 \text{ or } M = L\frac{R}{R + S} \qquad (10)$$

when

$$\omega t = \frac{\pi}{2}, P - \frac{M}{RC} = 0 \qquad \text{or} \qquad M = PRC$$

It is thus seen that there are two conditions which must be satisfied in order that there should be no deflection of the galvanometer, when an alternating E.M.F. is applied. The second of these is the same as for the original Carey-Foster circuit, which is obtained by putting $S = O$. The impossibility of satisfying the first condition under these circumstances is obvious for then $M = L$, an inflexible condition, difficult to satisfy. This circuit, while not strictly a bridge, resembles one in that two balance conditions are necessary.

134. Experiment 23. *Heydweiller's Method for Mutual Inductance.*—Connect the apparatus as shown in Fig. 101. For M use a pair of coils whose relative positions may be varied, and for C, a subdivided standard condenser. The purpose of the experiment is to determine M as a function of the setting of the movable coil. As source and detector use either the wire interrupter and vibration galvanometer, or an alternator and telephone. From the known E.M.F. of the source, compute the minimum value of R in order that the power consumption in it should not exceed 4 watts per coil.

Report.—Plot M as a function of the scale readings of the instrument. In computing M use the second of eq. (10). Check the accuracy of your results by substituting in the first of these equations and note the constancy of the values for L. In connecting up the circuit is there any choice as to which coil is used as the primary? Does the mutual inductance of two coils depend upon which is the primary? Explain.

135. Mutual Inductance by Heaviside's Bridge.[1]—If one of the arms of a Wheatstone bridge is inductive while the other three are non-inductive it is impossible to obtain a balance since the E.M.F. across the inductive arm will have a component 90 deg. out of phase with the current through it, and the E.M.F.'s at the galvanometer corners of the bridge can never be in phase. It was pointed out by Hughes that if in series with the galvanometer there is connected the secondary of a variable mutual

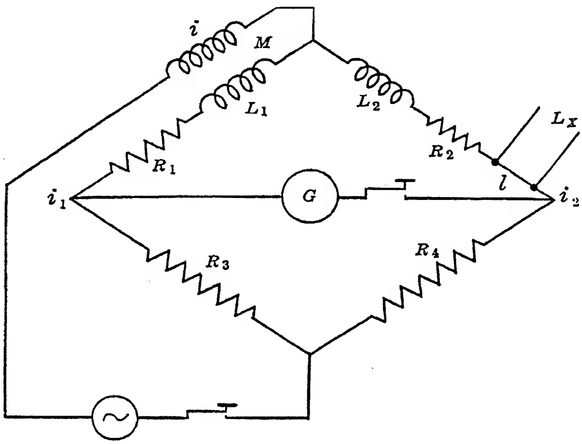

FIG. 102.—Heaviside's bridge for mutual inductance.

inductance, the primary of which is included in the supply circuit, an E.M.F. in quadrature with this current and hence opposite in phase to the E.M.F. due to the self inductance of the bridge coil may be obtained and a balance thus secured.

In discussing this circuit, Heaviside pointed out that a more satisfactory arrangement results if the secondary of the mutual inductance is introduced, not in the galvanometer circuit, but in the arm of the bridge adjacent to that containing the inductance under consideration. The E.M.F. thus induced in L_1 by mutual inductance may be made to compensate the difference in the E.M.F's of self inductance in L_1 and L_2. Such an arrangement is shown in Fig. 102. The balance condition is obtained as follows:

[1] *Phil. Mag.*, vol. 19, 1910, p. 497.
The Electrician, vol. 76, 1885–86, p. 489.

Designating by i, i_1, and i_2 the instantaneous supply and bridge currents respectively, the following equations result.

$$i = i_1 + i_2 \tag{1}$$

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di}{dt} = R_2 i_2 + L_2 \frac{di_2}{dt} \tag{2}$$

$$R_3 i_1 = R_4 i_2 \tag{3}$$

Eliminating i between eqs. (1) and (2), we have

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M\left(\frac{di_1}{dt} + \frac{di_2}{dt}\right) = R_2 i_2 + L_2 \frac{di_2}{dt} \tag{4}$$

Substituting in eq. (4) the value of i_2 from eq. (3)

$$R_1 i_1 + \left[L_1 + M\left(1 + \frac{R_3}{R_4}\right)\right]\frac{di_1}{dt} = \frac{R_2 R_3}{R_4} i_1 + L_2 \frac{R_3}{R_4}\frac{di_1}{dt} \tag{5}$$

Imposing now the condition for steady state balance, the terms in i_1 vanish, whence

$$L_1 + M\left(1 + \frac{R_3}{R_4}\right) = L_2 \frac{R_3}{R_4} \tag{6}$$

or

$$M(R_3 + R_4) = L_2 R_3 - L_1 R_4 \tag{7}$$

whence

$$M = \frac{L_2 R_3 - L_1 R_4}{R_3 + R_4} \tag{8}$$

If an equal arm bridge is used, i.e., $R_3 = R_4$

$$M = \frac{1}{2}\left[L_2 - L_1\right] \tag{9}$$

Campbell has suggested a simple modification of this bridge whereby self inductances may easily be measured in terms of mutual, provided a continuously variable standard of the latter is available. This inductance is introduced at l, shown short circuited by a link in the figure. A balance is first obtained with the link inserted. Let M_1 be the reading of the variable standard for this setting. Introduce the unknown by removing the link and balance again varying R_1 or R_2 to compensate for the added resistance of the unknown coil. Let M_2 be the new reading of the standard. Then, for an equal arm bridge,

$$M_1 = \tfrac{1}{2}(L_2 - L_1)$$
$$M_2 = \tfrac{1}{2}(L_2 - L_1 + L_x)$$

whence

$$L_x = 2(M_2 - M_1) \tag{10}$$

where L_x is the unknown self inductance to be measured.

A further simplification results if L_2 is a variable inductance, for then the first balance may be obtained by making M_1 zero and adjusting L_2 until it is equal to L_1. When a second balance has been obtained, L_x is simply twice the value of M. Neither L_1 nor L_2 need be known. This method is particularly useful where a number of inductances of the same order of magnitude are to be measured.

136. Experiment 24. *Heaviside's Bridge for Self Inductance.*—Connect the apparatus as shown in Fig. 102, using for M a variable standard of mutual inductance. L_2 should be a continuously variable self inductance. As a detector, use phones, vibration or A.C. galvanometer with appropriate source. Measure a series of self inductances.

Report.—Is there any choice, in this bridge, as to which of the two coils of the mutual inductance is used as the primary? May the leads to the primary be interchanged at liberty? Could a variable state balance be obtained if the unknown were introduced in the arm R_4? Explain.

137. Maxwell's Bridge for Mutual Inductance.[1]—The simplest, though not the most sensitive bridge for the measurement of mutual inductance is one devised by Maxwell. The method consists in obtaining the mutual inductance of a pair of coils in terms of the self inductance of one of them. The connections are shown in Fig. 103. In the discussion of the Heaviside bridge, Fig. 102, it was pointed out that a balance could be obtained by introducing in the coil L_1 by mutual inductance an E.M.F. which would compensate for the difference in E.M.F.'s in the coils L_1 and L_2. It might equally well have been said that the E.M.F. in the coil L_2 balances the difference between the E.M.F. in L_1 due to mutual and self inductance. If the relative values of the currents in the primary and secondary of M are changed these E.M.F.'s may be made equal without the use of the coil L_2. This is the method employed in the Maxwell bridge, and is accomplished by shunting the entire bridge by the resistance R.

Indicating the instantaneous currents as shown in the figure, the equations for the balance condition are as follows:

$$i = i_1 + i_2 + i_3 \qquad (1)$$

$$R_3 i_2 = R_1 i_1 + L\frac{di_1}{dt} - M\frac{di}{dt} \qquad (2)$$

$$R_2 i_1 = R_4 i_2 \qquad (3)$$

$$R i_3 = (R_3 + R_4) i_2 \qquad (4)$$

[1] Maxwell, Electricity and Magnetism, vol. 2, p. 365.

Eliminating i between eqs. (1) and (2)

$$R_1 i_1 + L\frac{di_1}{dt} - M\left(\frac{di_1}{dt} + \frac{di_2}{dt} + \frac{di_3}{dt}\right) = R_3 \tag{5}$$

But, from eq. (3)

$$i_2 = \frac{R_2}{R_4} i_1$$

and, from eq. (4)

$$i_3 = \frac{R_3 + R_4}{R} i_2 = \frac{R_3 + R_4}{R} \cdot \frac{R_2}{R_4} i_1$$

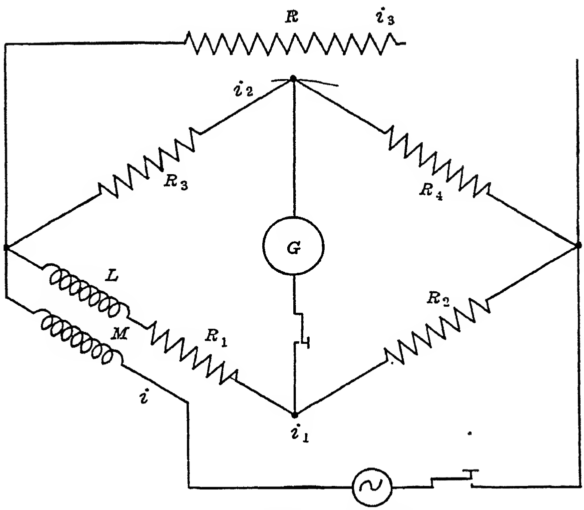

FIG. 103.—Maxwell's bridge for mutual inductance.

Substituting these values in eq. (5) there results

$$R_1 i_1 + L\frac{di_1}{dt} - M\left[1 + \frac{R_2}{R_4} + \frac{R_3 + R_4}{R} \cdot \frac{R_2}{R_4}\right]\frac{di_1}{dt} = \frac{R_3 R_2 i_1}{R_4} \tag{6}$$

Imposing the condition for a steady state balance, the terms in i_1 vanish, and we have

$$M\left[1 + \frac{R_2}{R_4} + \frac{R_3 + R_4}{R} \cdot \frac{R_2}{R_4}\right] = L \tag{7}$$

Since

$$\frac{R_1}{R_2} = \frac{R_3}{R_4},$$

we have

$$\frac{R_3 + R_4}{R_4} = \frac{R_1 + R_2}{R_2}$$

and eq. (7) may be written

$$M\left[1+\frac{R_2}{R_4}+\frac{R_1+R_2}{R}\right]=L \tag{8}$$

138. Experiment 25. *Mutual Inductance in Terms of Self Inductance by Maxwell's Bridge.*—Connect the apparatus as shown in Fig. 103 using for M several pairs of coils with fixed mutual inductances. Operate the bridge either with phones, vibration or A.C. galvanometer, and appropriate source of supply. After the determination of M for each pair of coils, interchange primary and secondary and check your result.

Report.—Is the balance as sharply defined as in some of the bridges previously used? Explain. At the balance point, are the currents in R and R_1 in phase?

139. The Mutual Inductance Bridge.—Figure 104 represents a bridge in which two coefficients of mutual inductance may readily be compared provided one of them is a variable standard. Designating the various parts of the bridge as indicated in the figure, the balance conditions are as follows:

$$R_1i_1+L_1\frac{di_1}{dt}+M_1\frac{di_1}{dt}=R_2i_2+L_2\frac{di_2}{dt}+M_2\frac{di_2}{dt} \tag{1}$$

$$R_3i_1-M_1\frac{di_1}{dt}=R_4i_2-M_2\frac{di_2}{dt} \tag{2}$$

Suppose that a variable state balance has been obtained with the secondaries disconnected and the galvanometer joined directly to the points A and B. This balance may be facilitated by the introduction of a variable inductance in series with either R_1 or R_2 as the case may require. Under these circumstances eqs. (1) and (2) become the same as those for the simple inductance bridge, namely

$$R_1i_1+L_1\frac{di_1}{dt}=R_2i_2+L_2\frac{di_2}{dt} \tag{3}$$

and

$$R_3i_1=R_4i_2 \tag{4}$$

whence

$$i_2=\frac{R_3}{R_4}i_1 \tag{5}$$

and

$$\frac{di_2}{dt}=\frac{R_3}{R_4}\frac{di_1}{dt}$$

Substituting in eq. (3) we have

$$R_1 i_1 + L_1 \frac{di_1}{dt} = \frac{R_2 R_3 i^1}{R_5} + L_2 \frac{R_3}{R_4}\frac{di_1}{dt} \tag{6}$$

Imposing the steady state balance, we have

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \tag{7}$$

and

$$\frac{L_1}{L_2} = \frac{R_3}{R_4}$$

Introduce now the secondary coils as shown in the figure and obtain a balance by adjusting the variable standard. This balance indicates that the E.M.F.'s in the secondary coils are equal and opposite. Since no current flows in the secondary coils, the currents for the primaries which are defined by eqs. (3) and (4) are unchanged. Accordingly, the values for i_2 and $\frac{di_2}{dt}$ given in eq. (5) may be substituted in eq. (1). Subtracting eq. (3) from eq. (1) then gives

$$\frac{M_1}{M_2} = \frac{R_3}{R_4} \tag{8}$$

This bridge is distinguished from those previously studied in that three balances are necessary. This may seem at first sight to result in an unduly cumbersome method, but experience shows that in reality it is a relatively simple bridge to operate.

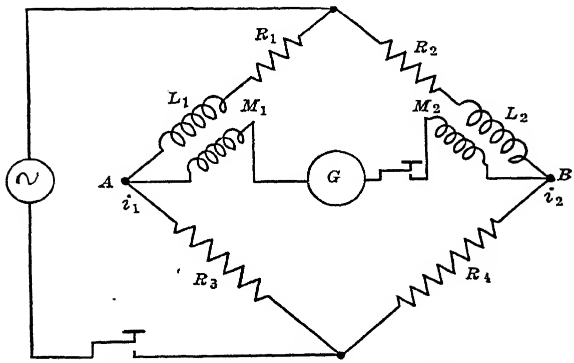

FIG. 104.—Mutual inductance bridge.

140. Experiment 26. *Comparison of Two Mutual Inductances.* Connect the apparatus as shown in Fig. 104, using a pair of phones and a suitable source of alternating current. Obtain the

three balance conditions as described above, for several different unknown pairs of coils. Check your results by interchanging primaries and secondaries for each pair.

Report.—Explain why it is permissible to introduce an extra inductance in series with R_1 and R_2. Could this be introduced in R_3 or R_4?

141. The Frequency Bridge.—In all of the bridges thus far discussed, the balance condition has, in no case, contained a term depending upon the frequency. The physical significance of this is that these bridges balance independent of the frequency and hence the form of the impressed wave is of little consequence. A bridge will now be studied which, for given values of L and C, can be balanced for only one definite frequency. The connections are shown in Fig. 105. Three of the arms are non-reactive, while the fourth contains an inductance and a condenser in series. Let the instantaneous currents through the upper and lower arms be i_1 and i_2 respectively. The balance conditions are then

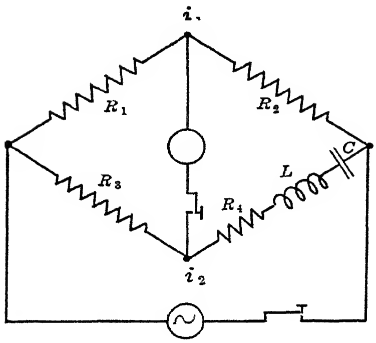

Fig. 105.—The frequency bridge.

$$R_1 i_1 = R_3 i_2 \tag{1}$$

$$R_4 i_2 + L\frac{di_2}{dt} + \frac{1}{C}\int i_2 dt = R_2 i_1 \tag{2}$$

Eliminating i_1, we have

$$R_4 i_2 + L\frac{di_2}{dt} + \frac{1}{C}\int i_2 dt \quad \frac{R_2 R_3}{R_1} i_2 \tag{3}$$

Imposing the steady state balance condition, which may be obtained by short circuiting C, in case a battery and ordinary galvanometer are used,

$$L\frac{di_2}{dt} + \frac{1}{C}\int i_2 dt = 0 \tag{4}$$

Assuming that a pure sine wave of E.M.F. is applied to the bridge, the current through it will also be a sine wave of the same frequency as the source. Let the current i_2 then be given by

$$i_2 = I \sin \omega t$$

Then

$$\frac{di_2}{dt} = I\omega \cos \omega t, \text{ and } \int i_2 dt = -\frac{I}{\omega} \cos \omega t$$

Substituting these values in eq. (4), we have

$$LI\omega \cos \omega t - \frac{1}{C\omega} I \cos \omega t = 0 \tag{5}$$

whence

$$\omega = \frac{1}{\sqrt{LC}}$$

Since $\omega = 2\pi n$, where n is the frequency,

$$n = \frac{1}{2\pi\sqrt{LC}} \tag{6}$$

When two of these quantities are known, the third may be computed. One of the most useful applications of this bridge is the measurement of frequency using a subdivided condenser and a continuously variable inductance. In case a complex wave is applied to the bridge, complete silence in the phones can not be obtained for any value of the LC product. It will be observed however, that the relative intensities of the fundamental and overtones will be changed as L is varied and that for a certain setting, the fundamental will disappear while the overtones remain.

In Art. **110**, it was pointed out that a circuit connected for parallel resonance, possesses a very large impedance for the particular frequency to which it resonates. If now such a circuit is placed in series with the source, and adjusted to resonate to the frequency of the fundamental as above determined, this frequency may be suppressed and that of the strongest overtone measured. Introducing now another resonance circuit in series with the source to suppress this overtone, the next stronger one may be measured and so on. In this way a qualitative analysis of the wave form of the source may be made.

142. Experiment 27. *Bridge Method for Measuring Frequency.* Connect the apparatus as shown in Fig. 105, using for C a subdivided mica condenser, and for L, a variable standard of inductance. Determine the frequency of several sources of alternating current, using the phones as a detector. Determine first the fundamental and then place a filter circuit in series with the source to suppress this frequency and measure the frequency for the strongest harmonic. In computing the frequency by eq.

(6), the inductance and capacitance must be expressed in henries and farads respectively.

Report.—Compute a constant for the right-hand side of eq. (6) which, when divided by $\sqrt{LC}$ will give the frequency when L is expressed in millihenries and C in microfarads. Compute the inductance, which, when used with a capacitance of 1 microfarad will balance the bridge for a frequency of 60 cycles.

143. Circuits of Variable Impedance.—In the bridges studied thus far for the measurement of self and mutual inductance, the assumption has tacitly been made that the only E.M.F.'s induced in the coils are those due to the primary current. For example, in the case of mutual inductance, a varying current in the primary produces an E.M.F. in the secondary proportional to the rate of change of the primary current hence in quadrature with the primary current for the case of a sine wave. For self inductance, the coil is its own secondary, and the same considerations hold as for two coils. The direction of the induced E.M.F. is counter to the driving E.M.F. while the current is rising, and in the same direction when it is falling. The power associated with the induced E.M.F. at any instant is equal to the product of this E.M.F. and the current. Energy accordingly is alternately stored in the electromagnetic field of the coil and returned to the circuit. The theory shows, in fact, that this occurs at a frequency twice that of the driving E.M.F.

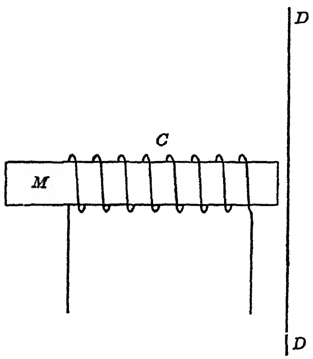

FIG. 106.—Simplified telephone receiver.

When the circuit is of such a nature that energy is consumed by the coil or parts connected with it in some other manner than by heat developed within the primary coil, the quadrature relationship is destroyed and the impedance of the coil is no longer constant. Among the more important causes of such extraneous energy consumption are hysteresis, eddy currents, and motion of parts. The telephone receiver is an illustration of such a circuit. For simplicity, consider a coil of wire C wound upon a bar magnet M near one end of which is placed a flexible iron diaphragm D as shown in Fig. 106. When an alternating current flows through

C, the residual magnetism is alternately increased and decreased by the current and the diaphragm vibrates with the same frequency as the source. In addition to the Joule heat, represented by I^2R, developed in the coil, energy consumptions result from the three causes enumerated above in the following manner.

1. *Hysteresis.*—The E.M.F. induced in the coil is proportional to the rate of change of the magnetic flux through it. This flux is produced by the magnetomotive force of the coil, the latter being in phase with the current and proportional to it. Because of the hysteretic lag of flux behind the magnetomotive force, the E.M.F. is no longer in quadrature with the current but has a component counter to the current which results in a continuous energy consumption. The greater the area of the hysteresis loop, the greater the lag of flux behind the current and hence the larger the energy component of the induced E.M.F. The hysteresis loss is proportional to the frequency.

2. *Eddy Currents.*—The magnet M may be regarded as a secondary coil consisting of a single turn about its own axis having a relatively large cross section and a low resistance. The changing flux through this turn induces in it an E.M.F. in quadrature with the flux and the resulting current is known as a Foucault or eddy current. Because of the small self inductance of this single turn, the eddy current is practically in phase with the induced E.M.F. producing it. The eddy current may in turn be considered as a primary which induces in the coil a quadrature E.M.F. Except for the hysteresis lag, this final E.M.F. in C is counter to the current because of the double quadrature relationship, and hence introduces a large energy consumption. Viewed from the standpoint of Joule heat developed in the core by the eddy current, this loss is proportional to the square of the frequency, for the induced E.M.F. producing the eddy current is proportional to the frequency and the heating effect of a current is equal to the square of the E.M.F. divided by the resistance.

3. *Motion of the Diaphragm.*—The effect of motion of the diaphragm may be understood by the following considerations. Suppose a sound wave strikes the diaphragm. The varying air pressures cause it to vibrate and in so doing, the air gap between it and the magnet is changed and hence the reluctance of the magnetic circuit of the magnet. This introduces a change in flux through the coil which induces an E.M.F. within it. In

fact this is the principle of the "magneto-phone" which is often employed where accurate reproduction is more essential than energy delivered. As regards the magnitude of the E.M.F. induced in the coil and the phase relation between it and the motion of the diaphragm, it makes no difference whether the motion is produced by a sound wave or by a current through C. In the latter case, the energy of the wave must be supplied by the current, and the law of conservation of energy requires that the induced E.M.F. due to the motion of the diaphragm must have a component counter to the current to account for this consumption.

If an alternating current of intensity I is passed through the coil, and the power delivered to the coil is measured by appropriate means, it is found that this is much larger than would be computed from I^2R when R is determined by using direct current. On the other hand we may define a resistance R_e such that

$$\text{Watts} = I^2R_e.$$

R_e is called the "effective" resistance of the coil. It is the resistance of a fictitious coil, free from hysteresis, eddy-current and motional reactions, which consumes the same power with a given current. Again the effective resistance may be written

$$R_e = R + R_H + R_E + R_M$$

where the last three terms represent the parts contributed by hysteresis, eddy currents and diaphragm motion respectively, and, it is customary to speak of the resistance due to hysteresis, eddy currents, etc. In a similar manner, the E.M.F.'s induced in the coil by hysteresis, eddy currents and motion will have components in quadrature with the current, and will change the apparent inductance of the coil, and it is customary, in an analogous manner, to speak of the inductance due to hysteresis, eddy currents, etc.

Kennelly and Pierce[1] have made a detailed study of the motional characteristics of telephone receivers and have shown how their performance in practice may be predetermined from simple measurements. The receiver to be studied was placed in one of the arms of an inductance bridge and its effective resistance and inductance measured for a wide range of frequencies, first with the diaphragm clamped, and again when free to move.

[1] *Proc. Am. Acad. of Sci.*, vol. 48, p. 131, 1912.

The difference between the corresponding values for the same frequency were called "motional resistance" and "motional inductance" respectively. The latter when multiplied by the frequency for which they were determined, gave the "motional reactance" for that frequency. Interesting results were obtained for frequencies near that corresponding to the natural period of the diaphragm. For example, the curve showing the motional resistance as a function of the frequency closely resembles, near the resonance frequency, the curve in optics, showing the variation of the index of refraction with frequency in the neighborhood of an absorption band, while that for motional reactance exhibits a sharp minimum at this point.

144. Experiment 28. *Motional Impedance of a Telephone Receiver.*—Connect the apparatus as shown in Fig. 64 substituting for L_x the receiver to be studied. Use an equal arm bridge making R_1 an R_2 approximately equal to the direct current resistance of the receiver. Energize the bridge with a Vreeland oscillator which has previously been calibrated for frequency, and use a pair of head phones as a detector. Place an electrostatic voltmeter across the output coil of the oscillator and maintain a constant voltage on the bridge throughout the experiment. Determine roughly the natural period of the diaphragm of the receiver by varying the frequency of the oscillator keeping the voltage approximately constant by noting at what frequency the response is loudest. Introduce a small wedge between the diaphragm and the cap to prevent motion and measure the resistance and inductance for a range of frequencies above and below the resonance frequency. Remove the wedge and repeat with the diaphragm moving.

Report.—Plot curves showing the variation of resistance and reactance with frequency for both blocked and moving diaphragm. Subtract the former from the latter and thus obtain the "motional" resistance and reactance and plot each as a function of frequency.

145. Power Factor and Capacitance of Condensers.[1]—In a perfect condenser, that is, one without absorption or leakage, the phase of the current is 90° ahead of the E.M.F. impressed across its terminals. Although many condensers approximate the ideal, it is only with well insulated air condensers that this condition may

[1] Grover, Bull. U. S. Bureau of Standards, vol. 3, 1907, p. 371.
Wien, *Wiedemann's Annalen*, vol. 44, 1891, p. 689.

be regarded as actually realized. In condensers having a dielectric made of paper impregnated with parafine or beeswax there is an appreciable component of the current in phase with the E.M.F. In such a condenser there is a measurable amount of energy absorption which appears as heat in the dielectric, and as far as phase relations are concerned, it may be regarded as a perfect condenser with a small ficticious resistance in series with it. In Fig. 107*a*, let C represent the equivalent perfect condenser, and ρ the fictitious series resistance. The vector diagram 107 *b* represents the phase relations for such a circuit. OE is the

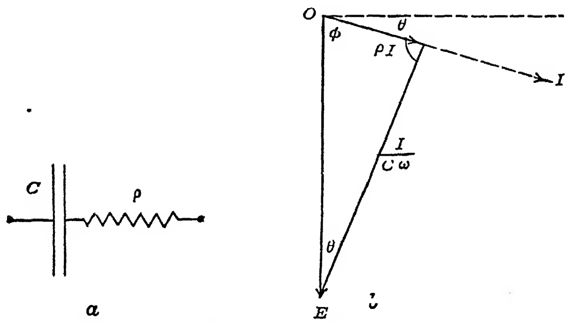

FIG. 107.—Phase diagram for a condenser.

impressed E.M.F. and OI the current which falls short of the 90° lead by the angle θ which is designated as the phase difference of the condenser. ϕ is the phase angle as ordinarily defined and the power factor is then

$$\text{P.F.} = \cos \phi = \sin \theta.$$

It is obvious from the figure that

$$\tan \theta = \rho C\omega \tag{1}$$

A simple bridge method has been devised by Wein by which both the capacitance and the power factor of an imperfect condenser may be simultaneously measured provided there is available for comparison purposes another condenser which shows no absorption. Such a bridge is illustrated in Fig. 108, where C_1 is the perfect condenser and C_2 the one with ficticious resistance ρ to be studied. In series with these condensers are placed the small finely adjustable resistances r_1 and r_2. The purpose of these is to bring about equality of phase in the currents through the upper and lower branches of the bridge. It is clear that,

without them, if one of the condensers possesses an equivalent resistance while the other does not, the potential differences from D to B and E to B can not be equal and in phase at the same time. Accordingly a perfect balance of the bridge can not be obtained. If, however, a suitable resistance r_1 is introduced in series with C_1 of such a value that the time constant of the arm DB equals that of EB, this difficulty is obviated. In practice it is generally more convenient to introduce r_2 also and take account of it in deducing the balance conditions.

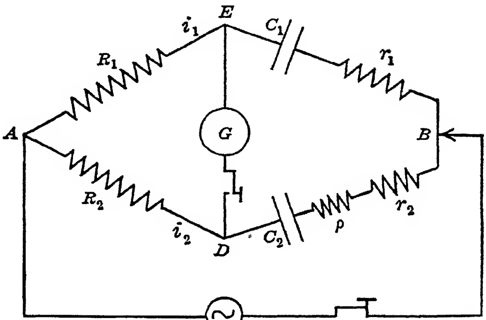

FIG. 108.—Bridge for measuring phase difference of a condenser.

The equations for balance may be derived in the following manner, calling i_1 and i_2 the instantaneous currents in the upper and lower arms respectively.

$$R_1 i_1 = R_2 i_2 \tag{2}$$

$$\frac{1}{C_1}\int i_1 dt + r_1 i_1 = \frac{1}{C_2}\int i_2 dt + (\rho + r_2) i_2 \tag{3}$$

Eliminating i_2 we have

$$\frac{1}{C_1}\int i_1 dt + r_1 i_1 = \frac{R_1}{R_2}\frac{1}{C_2}\int i_1 dt + (\rho + r_2)\frac{R_1}{R_2} i_1 \tag{4}$$

Imposing the condition for a steady balance, namely

$$\frac{R_1}{R_2} \quad \frac{}{r_2 + \rho} \tag{5}$$

there results

$$\frac{C_1}{C_2} = \frac{R_2}{R_1} \tag{6}$$

The phase difference θ may be obtained as follows:

Combining eqs. (5) and (6) we have

$$\frac{C_1}{C_2} = \frac{r_2 + \rho}{r_1} \tag{7}$$

Multiplying numerator and denominator on the left by ω and clearing of fractions, there results

$$C_1\omega r_1 = C_2\omega(r_2 + \rho) \tag{8}$$

Referring to Fig. 107*b* and solving eq. (8)

$$\tan\theta = C_2\omega\rho = C_1\omega r_1 - C_2\omega r_2 \tag{9}$$

145A. Self-contained Capacitance Bridge.—For the precise measurement of small capacitances or the determination of their dielectric losses the ordinary type of bridge described in the previous section is unsatisfactory since the stray capacitances in the various parts of the circuit are of the same order of magnitude as the capacitance to be measured. A bridge for the measurement of capacitances of the order of those used in radio work, i.e., from very small values up to several hundred micro-microfarads, requires complete shielding of all its parts.

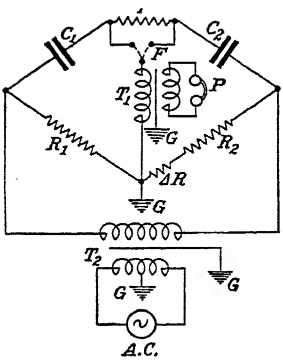

FIG. 109.—Circuit for General Radio capacitance bridge.

The General Radio Company has developed a convenient self-contained bridge for this purpose, the elementary circuit for which is shown in Fig. 109. It is similar to that of Fig. 109 except that the source of A.C. supply and the detecting device are interchanged. The ratio arms R_1 and R_2 are 5,000 ohms each, so the bridge is always used with a one-to-one ratio. The resistance r is a four-dial decade box which may be placed in series with either condenser C_1 or C_2 by the switch F. Power is supplied through the transformer T_2, and the phones P serve as the detecting device. Since the small condensers C_1 and C_2 have impedances at audio frequencies much higher than the ordinary telephone receiver, a step-down transformer is used to secure a better impedance match between bridge and phones.

It is often desirable to calibrate a vernier condenser whose capacitance is of the order of a few micro-microfarads. This may be accomplished as follows: Use for C_1 and C_2 equal capacitances of about 1,000 micro-microfarads and secure a balance.

Then place the vernier condenser in parallel with C_1. If a resistance ΔR of 5 ohms, or one part in one thousand, is added to R_2, then when the vernier condenser is set at one micro-micro-

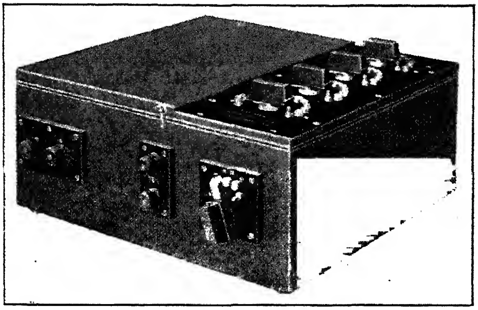

Fig. 110.—General Radio capacitance bridge.

farad, a balance again results. The bridge is provided with three ΔR units to give unbalances of one part in one thousand, one in one hundred, and one in ten respectively.

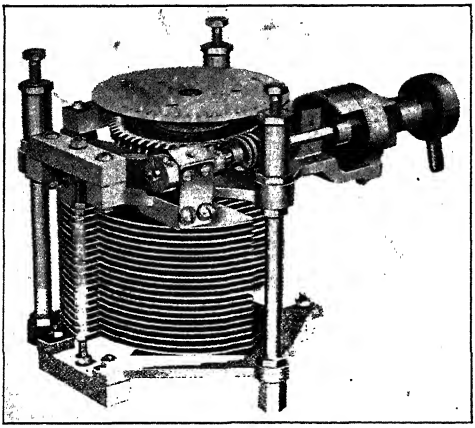

Fig. 111.—General Radio precision variable air condenser.

All the elements of the bridge except the condenser are placed in a wooden box with a copper lining, and to secure isolation of one element from another, each is placed in a separate compartment with copper partitions. The transformers T_1 and T_2 have

grounded copper shields between their primary and secondary windings, the former to prevent errors due to the capacity between the observer's body and the bridge, and the latter to protect against variable capacities to source of current supply. The instrument, completely assembled, is shown in Fig. 110. A precision air condenser suitable for use in this bridge is shown in Fig. 111. The plates are of heavy aluminum separated by accurately turned spacers and are firmly clamped between substantial metal end plates. A steel shaft, carrying the rotating plates, turns in cone-shaped bronze bearings. This shaft is rotated by a worm and gear, thus permitting fine control of settings, and giving vernier readings. The worm is held in position against the gear by a spring which thus prevents back lash and secures easy operation. Special care has been used to reduce the volume of the insulating material between the fixed and movable plates to a minimum and to place it in a region of small electric field. In this way a small power factor and consequently small-phase difference, i.e., angle θ Fig. 107, has been secured.

146. Experiment 29. *Measurement of Phase Difference and Capacitance of a Condenser.*—Connect the apparatus as shown in Fig. 108. C_1 is a standard mica condenser whose phase difference is regarded as negligible, and C_2 is a telephone condenser with paraffine paper dielectric whose phase difference and capacitance are to be determined as a function of the frequency. The resistances r_1 and r_2 are small in value and should be joined by a slide wire for fine adjustment. As a source of power use an oscillator giving a pure wave form whose frequency may be varied over a considerable range, such as the Vreeland, with phones as detector. In obtaining a balance, set $r_2 = 0$ and get as good silence as possible. Introduce such a value of r_1 as makes the best improvement, then change R_1 or R_2 and again adjust r_1 and so on until complete silence is reached. Keep r_2 as small as possible. Make a series of balances using as wide a range of frequencies as may be obtained from the oscillator.

Report.—Plot capacity and phase difference of the unknown condenser as a function of the frequency. Show that in a perfect condenser the current leads the E.M.F. by 90°. Define *Power Factor.*

147. Resistance of Electrolytes.—The measurement of the resistance of an electrolyte offers special difficulties not encoun-

tered in determining the resistance of metallic conductors. This is due to the fact that current is carried through a solution by virtue of the migration of ions, a double procession in opposite directions. These are deposited on the electrodes, where secondary chemical reactions often take place. In general, the deposits on the electrodes set up counter E.M.F.'s in the cell which affect the measurements in the same manner as added resistance. Obviously then an electrolytic resistance can not be measured by a Wheatstone bridge employing direct currents. If an alternating current is used this effect is eliminated since the counter E.M.F. is with the bridge current during one half of the cycle and opposite to it during the other.

In case an electrolyte is measured in which a gas is formed at one of the electrodes a further complication is introduced since

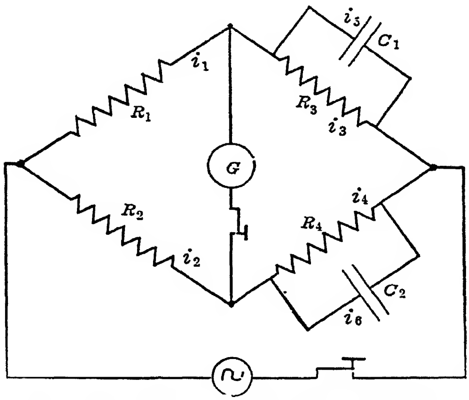

FIG. 112.—Bridge for electrolytic resistance.

the cell behaves as though it contains capacitance. This results from the fact that a gas layer separates the liquid from the electrode thus forming a condenser. Since the gas layer is, in general, very thin, a capacitance of considerable magnitude may result. The cell then behaves like a condenser and resistance in parallel, and it must be so regarded when connected in one of the arms of a bridge. The resistance in the adjacent bridge arm must also be shunted by a condenser else a balance can not be obtained. Such a bridge is shown in Fig. 112, where R_4 and C_2 represent the resistance and capacitance of the electrolytic cell,

and R_3 and C_1 its counterpart in the adjacent arm. Designating the currents as indicated in the figure, we have

$$i_1 = i_3 + i_5 \tag{1}$$

$$i_2 = i_4 + i_6 \tag{2}$$

$$R_1 i_1 = R_2 i_2 \tag{3}$$

$$R_3 i_3 = R_4 i_4 \tag{4}$$

$$\frac{1}{C_1}\int i_5 dt = R_3 i_3 \tag{5}$$

$$\frac{1}{C_2}\int i_6 dt = \tag{6}$$

Eliminating i_5 between eqs. (1) and (5) and i_6 between eqs. (2) and (6) there results

$$i_1 = i_3 + C_1 R_3 \frac{di_3}{dt} \tag{7}$$

$$i_2 = i_4 + C_2 R_4 \frac{di_4}{dt} \tag{8}$$

Substituting in eq. (8) the values of i_2 and i_4 from eqs. (3) and (4) and eliminating i_1 between the resulting equation and eq. (7) we have

$$i_3 + C_1 R_3 \frac{di_3}{dt} = \frac{R_2}{R_1} \cdot \frac{R_3}{R_4} i_3 + C_2 R_3 \frac{R_2 di_3}{R_1 dt} \tag{9}$$

Imposing now the condition for steady state balance, *i.e.*,

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \tag{10}$$

we have

$$\frac{C_1}{C_2} = \frac{R_2}{R_1} \tag{11}$$

In carrying out measurements of the resistance of solutions, the design of the electrolytic cell is a matter of considerable importance. It has been found that different electrolytes require different types of cells and even for the same electrolyte a given cell is not always suited to wide ranges of concentration. For example, polarization may occur in some cases unless the electrodes are platinized, and in other cases platinized electrodes appear to act calclytically and assist chemical action. Again platinized electrodes may, because of their spongy nature, absorb so much of the electrolyte as to cause errors in measurement when used later with solutions of a different nature or concentration.

Figure 113 shows a cell designed by Dr. Washburn and manufactured by the Leeds and Northrup Co. The electrodes are of platinum and are mounted by sealing their supporting wires into tubular glass stems. These wires project through the seals and connections with them are made by filling the stems with mercury. Side tubes, above and below the electrodes respectively, are attached for filling and washing out the cell. These tubes are bent so as to form supports for holding the cell in a suitable bath for maintaining a constant temperature.

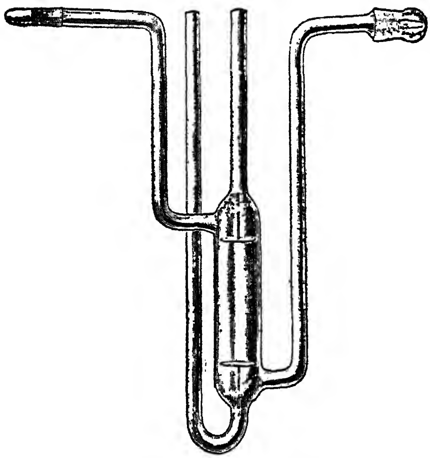

FIG. 113.—Cell for measurement of electrolytic resistance.

148. Experiment 30. *Resistance of Electrolytes.*—Connect the apparatus as shown in Fig. 112, placing the solution in a cell specially designed for the purpose. Energize the bridge with the Vreeland oscillator and detect the balance with a telephone receiver. Determine the resistance of a series of solutions furnished by the instructor. Measure the dimensions of the cell and the distance between electrodes and compute the specific resistance of each solution.

Report.—Explain why a bridge can not be balanced using direct currents when it contains an electrolytic cell. What is the essential difference between metallic and electrolytic conduction?

CHAPTER XIII

CONDUCTION OF ELECTRICITY THROUGH GASES[1]

149. Electrons.—When a high tension discharge passes between electrodes sealed into a partially evacuated vessel, the gas becomes luminous showing a series of highly colored glows which are often very beautiful. If the pressure is sufficiently reduced, a series of streams appears, proceeding in straight lines from the cathode. These streams are known as "cathode rays," and are found to be independent of the position of the anode, and often penetrate regions occupied by other glows in the tube.

The researches of modern physics have shown that these rays are streams of discreet particles of negative electricity, called "electrons." Their properties do not depend upon the material of the electrodes nor the nature or pressure of the gas through which the discharge takes place. They may be produced from all chemical substances, and consequently must play an important part in the structure of matter. The velocities with which they move through the tube vary from one-thirtieth to one-third that of light. The ratio of the charge of an electron to its mass is constant and is equal to 1.77×10^7 electromagnetic units per gram. The charge of an electron is 1.5×10^{-20} electromagnetic units and the mass is about $\frac{1}{1,830}$ that of the hydrogen atom. The radius of an electron is estimated, at 1.9×10^{-13} cms., which is about $\frac{1}{50,000}$ that of the atom. For many years the mass has been regarded as purely electromagnetic in character; that is, while exhibiting inertia, it shows no gravitational attraction in the sense possessed by ordinary matter. Recently, however, certain experimental and theoretical evidence has been produced which makes it appear likely that this cannot be entirely the case.

[1] Crowther, Ions, Electrons and Ionizing Radiations.
McClung, Conduction of Electricity through Gases and Radioactivity.
Millikan, The Electron.
Thomson, Discharge of Electricity through Gases.
Townsend, Electricity in Gases.

Many attempts have been made to discover evidence of quantities of electricity smaller or larger than the electron, but none smaller have ever been found. In fact, when quantities comparable to the electron have been isolated, they have always proved to be exact integral multiples of it. The evidence points to the conclusion that electricity is atomic in structure and that the smallest possible element is the electron, which thus constitutes our natural unit of electricity. Electric currents through conductors, as we know them in every day practice, are simply streams of electrons through or between the atoms and molecules making up the conducting body.

150. Conductivity of Gases.—A gas in its normal state is one of the best insulators known. This may be shown by mounting a gold leaf electroscope inside an inclosed space, and allowing only a small rod carrying a polished knob, for the purpose of charging, to project out. If the support carrying the electroscope is well insulated from the container, the electroscope will remain charged for a long time, showing that the air or whatever gas surrounds the electroscope is a poor conductor of electricity.

If, however, X-rays are allowed to shine through the enclosure, or if a small quantity of some radio-active substance such as thorium or radium is placed inside it, or again if the products of combustion of a flame are drawn through it, it is then found that the gold leaves collapse quite rapidly, indicating that the gas has lost its insulating properties. That the leakage has taken place through the air and not across the insulating support may be shown by using a second chamber connected with the electrometer enclosure by a glass tube, and introducing the X-rays, the radio active substance or other agent into this, and then drawing the air thus acted upon into the first chamber. The same effects are observed. However, if glass wool is introduced in the connecting tube, or if the air is passed between two insulated plates connected to a battery before entering the electrometer chamber, it is found that its insulating properties are restored. Experiments of this sort as well as many others of an entirely different nature have shown that the conduction of electricity through gases is due to carriers of electricity, and that the carriers are of two distinct types, positive and negative; the former are similar to the carriers of electricity through solutions and are called positive ions, while the latter are either negative ions or electrons.

151. Structure of the Atom.—To explain the phenomena of the conductivity of gases it is necessary first to make a brief statement concerning the structure of the atom. While our knowledge is far from complete, it is well established that the atom consists of a positive nucleus about which are placed electrons in some definite manner, the number and arrangement being characteristic of each chemical element. Except for hydrogen, the nucleus contains both positive elements of charge called "protons" and electrons. The proton is much smaller in size than the electron and has a correspondingly greater mass. In fact the entire measurable mass of the atom is attributed to the protons. The charge of the proton is numerically equal to that of the electron though of opposite sign. The neutral atom contains equal numbers of protons and electrons.

Approximately half of the electrons are associated with the protons in the nucleus while the others are distributed outside in a manner similar to that of the planets of the solar system about the sun. The experimental evidence seems to point to the fact that these extra-nuclear electrons are rotating about the nucleus.

Bohr, in applying the quantum theory to the atomic structure problem, assumes that there are definite orbits which the electrons can occupy—orbits in which the angular momentum of the electrons has definite values. When an electron occupies such an orbit the atom is said to be in a stationary state. This does not mean that the electron is stationary but rather that while the electron remains in the orbit the energy of the atom does not change. Thus we can say that the atom can exist in discreet energy states or "energy levels," each level being characterized by a definite potential and kinetic energy of the electron.

When the electron is removed entirely from the atom, the atom occupies the highest energy state and we have ionization.

Atoms may be lifted from any level to a higher one by absorption of radiation, or by collision with ions, alpha particles, or other electrons. When the atom changes from a higher energy state to a lower one, light is given off. The energy of this emitted light is equal to the difference in energy of atom in the two states, and the frequency of the light is proportional to the energy difference.

When external agencies such as X-rays, ultra-violet light, radiations from radio active materials, etc., act upon a gas, some

of the atoms are ionized; that is, one or more electrons are removed from the atoms by the absorption of energy from the incident radiation. We thus have present in the gas positive ions and negative electrons. The means by which this condition is brought about is called the "ionizing agent." If two electrodes are introduced, and a difference of potential is maintained between them, the electrons move to the positive electrode, and, entering it, pass on through the external metallic circuit. The positive ions, on the other hand, move to the negative electrode and receive electrons from it, thus becoming again neutral molecules. Unless an ionizing agent acts continuously, the current through the circuit will persist only until the ions and electrons have been removed from the gas.

152. The Ionization Current.—Suppose now that an ionizing agent is acting continuously upon a gas in an ionization chamber, as an arrangement such as that just described is called. At first it might be supposed that if the agent acts long enough all of the atoms would be ionized. This, however, is not the case; for, due to their undirected heat motion, ions and electrons collide, and recombine. When the rate of recombination is equal to that of ionization, a steady state is reached where only a definite fraction, usually a very small number, of the total number of molecules are in the ionized state. If the difference of potential between the plates is varied, and the current between them is measured and plotted as a function of voltage, it is found that the current increases with the voltage almost linearly at first, in accordance with Ohm's law; but for higher voltages, the curve is concave downward and when a certain voltage has been reached, no further increase in current can be obtained, unless the voltage is raised to very large values. The constancy of the current is due to the fact that all of the ions and electrons produced are swept out by the field. This current is spoken of as the "saturation current," from the similarity between the shape of this curve and the magnetization curve for iron. The voltage at which the horizontal part of the curve begins is called the "saturation voltage."

If the distance between the electrodes is increased, it might, by analogy with metallic conductors, be thought that the saturation current would be reduced because of the increased path the ions and electrons must travel. It is found, however, that the current is actually increased. This is because there is a larger

number of gas molecules subjected to the action of the ionizing agent, and hence more carriers are produced. Again, it is found that if the pressure of the gas is increased, the ionization current is increased. Both of these facts show that the saturation current through a gas depends on the mass of the gas between the electrodes.

153. Ionization by Collision.—If the voltage between the plates of the ionization chamber is increased to sufficiently large values, the saturation current does not remain constant indefinitely, for fields may be reached at which the current again begins to rise, slowly at first and then very rapidly, finally resulting in a disruptive spark accompanied by the passage of a current of considerable magnitude. The field required for this increased current depends upon the distance between electrodes, their size and shape, and the nature and pressure of the gas. For air at atmospheric pressure and spherical electrodes of moderate dimensions, e.g., 1 cm. diameter, it is of the order of 10,000 volts per centimeter. It diminishes, however, as the pressure is reduced, and is most conveniently studied at pressures below 10 millimeters of mercury.

This increase in current is due to the fact that ions are produced by collisions taking place between neutral molecules and ions as well as electrons already existing in the gas. The mechanism of this process is somewhat obscure, but it is clear that a definite amount of energy is required to disrupt a neutral atom. The kinetic energy of motion of the ions and electrons depends upon how far they have moved under the accelerating field before being stopped in the same way that the energy of motion of a freely falling body depends upon the distance through which it has fallen before being arrested. Thus, as the pressure of the gas is reduced, the average length of free travel is greater and the acquired energy available for ionizing purposes is increased. The conductivity of a gas therefore increases as the pressure is reduced. Since, however, the conductivity depends upon carriers which come originally from neutral molecules, the conductivity can not increase indefinitely with decrease of pressure, for the effect of the decreased available supply will eventually be felt. An optimum pressure therefore exists at which the increased range for acceleration is just balanced by the decreased supply of molecules. For air, this pressure is of the order of a few tenths of a millimeter of mercury. A further decrease in the

pressure results in a rapid increase in the resistance of the gas. If a perfect vacuum could be obtained, the free space between electrodes would be a perfect insulator. While this is, of course, impossible, it is, nevertheless, easy with modern methods of evacuation to obtain pressures so low that no appreciable discharge can be detected.

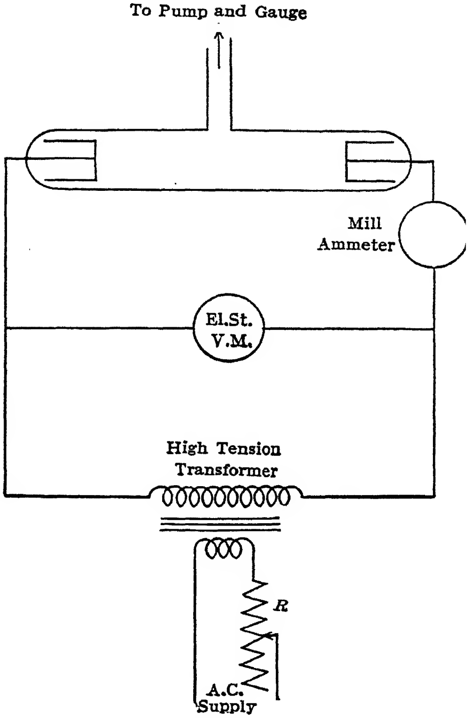

FIG. 114.—Resistance of discharge tube.

154. Experiment 31. *Resistance of a Discharge Tube.*—The apparatus consists essentially of a discharge tube, as shown in Fig. 114, about fifteen inches in length, through the ends of which are sealed wires attached to electrodes of relatively large area. It is connected to a high vacuum pump by means of which the pressure may be reduced to any desired value. A manometer and McLeod gauge are also jointed to the tube to measure the pressure.

Connect a small high tension transformer across the tube to supply the voltage for the discharge. Place an electrostatic voltmeter across the tube and an A.C. milliameter in series with it. The impressed voltage may be controlled by a series resistance in the primary circuit. Starting at one atmosphere, reduce the pressure until a current of 10 or 15 milliamperes is obtained through the tube. Measure the required voltage. Take a series of readings at various pressures measuring the voltage required to maintain a definite predetermined current. Compute the resistance of the tube by Ohm's law. Repeat the experiment using twice this current.

Report.—Plot a curve showing the resistance of the tube as a function of pressure. Why must the current be held constant in this experiment? Explain the operation of the McLeod gauge.

155. Phenomena of the Discharge Tube.—If electrodes are mounted at the ends of a tube such as shown in Fig. 115, con-

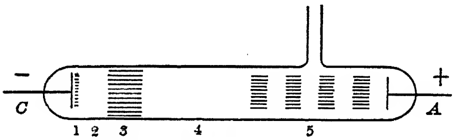

FIG. 115.—Luminous regions of discharge tube.

taining air at ordinary pressures and a sufficiently high voltage is impressed between them, the phenomenon first observed is the ordinary spark similar to that between the electrodes of a static machine. If, however, air is gradually removed, the sparks become less violent, and fine streamers of pinkish color are observed. As the pressure is further reduced, these streamers broaden out and fill the entire tube, and a pink color appears at the anode. With further exhaustion, the pink color extends some distance from the anode and dark spaces appear in the region of the cathode. When the pressure has been reduced to about half a millimeter of mercury, the discharge assumes a very characteristic appearance. Closely surrounding, but not quite touching the cathode, is a thin layer of luminosity known as the *cathode glow*. Next to this is a region, from which no light is observed, called the *Crooke's dark space*, and beyond this is a rather broad region of luminosity known as the *negative glow*. Following this is another non-luminous region, called the *Faraday dark space*. Between this dark space and the positive electrode

is a region called the *positive column*, which may be seen as a continuous band of light or, under certain conditions of current and voltage, as a series of light and dark striae. The positive column seems to be definitely associated with the anode, for if the tube is increased in length or bent into a curve, the positive column increases or bends with it, while the other parts of the discharge remain fixed and are thus shown to be associated with the cathode. Three luminous regions are indicated in Fig. 115.

If the pressure is still further reduced, the striae of the positive column become fewer in number and wider in extent and finally disappear. The regions associated with the cathode also become larger and, with the disappearance of the positive column, the dark spaces fill nearly the entire tube. With sufficient exhaustion, the Crookes dark space completely fills the tube, and the voltage required for a passage of current becomes very high. At this stage, the walls of the tube fluoresce brilliantly with colors depending upon its chemical composition, being bluish for soda, and bright green for German glass. If the exhaustion is carried far enough, the tube becomes a non-conductor of electricity.

156. Theory of the Discharge.[1]—Since no external ionizing agent is acting, it is obvious that the discharge is maintained by ions produced by collision, and the varied distributions of the luminous regions indicate that the electric fields and the velocities of the carriers can not be uniform throughout the tube. It has not yet been definitely determined whether luminescence arises from ionization of neutral molecules or whether it accompanies the recombination of an ion and an electron to form a neutral molecule. At the present time, the evidence seems to favor the latter hypothesis. Another widely accepted view is that when a molecule has been shaken up by collision with an ion or electron to such an extent that its electronic orbits are badly distorted, but not disrupted, light emission accompanies its return to the equilibrium state. On the latter theory, luminous regions do not necessarily coincide with regions of ionization. Some of the more important phenomena characterizing the several regions enumerated above are the following.

1. *Cathode Glow.*—The field strength in this region is large and often the greater part of the entire potential difference occurs in

[1] Crowther, Ions, Electrons, and Ionizing Radiations, chap. VI. Townsend, Electricity in Gases, chap. XI.

this limited space. The magnitude of the fall in potential depends upon the nature of the gas and the material of the electrode, ranging from 470 volts for water vapor to 170 volts for argon with platinum electrodes. If metals such as magnesium, sodium, or potassium are used, much smaller values are obtained because of the greater ease with which these substances emit electrons. The large potential gradient here is caused by the accumulation of positive ions in front of the cathode. Because of the greater mobility of electrons, they rapidly move away from this region thus leaving a preponderance of positive ions. The ionization is caused by collision of the positive ions either with gas molecules or the cathode itself.

2. *Crookes Dark Space.*—It was pointed out above that a certain amount of energy is required to produce ionization. The electrons from the cathode glow must move through a certain difference of potential before they possess the requisite kinetic energy for this purpose. The Crookes dark space represents this distance for it is here that electrons, liberated in the cathode glow, are acquiring the necessary energy of motion to produce the ionization of the negative glow. It is, in general, a rough measure of the mean free path of the electrons. No ionization occurs in this region and the current is carried almost exclusively by the electrons.

3. *Negative Glow.*—The luminosity of this region is due to ionization by electrons from the Crookes dark space. The positive ions produced here move slowly out of the negative glow into the Crookes dark space and their presence reduces the potential gradient to such an extent that electrons, originating in the negative glow, do not gain sufficient speed to produce ionization; and hence, after those entering from the Crookes dark space have been stopped by the ionization process, no further ionization occurs.

4. *Faraday Dark Space.*—The current in this region is due largely to electrons which enter it from the negative glow. Because of the accumulation of electrons in the negative glow, the potential gradient through the Faraday dark space and even up to the anode is quite large. The electrons are accordingly accelerated through this dark space and when they have attained velocities sufficient for ionization, the positive column commences.

5. *Positive Column.*—The potential gradient is practically constant throughout this region and ionization by collision may take place all the way, resulting in a uniform column of light.

Ordinarily, however, there are local accumulations of positive ions, which result in a decrease in the potential gradient with a consequent reduction in the acceleration of the electrons. There are then regions in which the velocities are too small to produce ionization and the striae commonly observed, result. Under these circumstances, the positive column is, to a certain extent, a repetition of the phenomena of the Crookes dark space, and the negative glow.

157. Investigation of the Field Strength at Various Points in the Discharge.[1]—The potential at any point in a tube may be determined by inserting an auxilliary electrode. A small platinum wire is most frequently used for this purpose. If the region happens to be one of high potential, the wire will attract to it positive ions until its potential is the same as that of its surroundings, which is then indicated by an electrometer to which the wire is attached. Accurate results can be obtained by this method only when there is a plentiful supply of ions of both signs. For example, suppose the wire is introduced near the anode, where only electrons are present. The forces of the field will cause electrons to strike the wire until it is so highly charged negatively that no more can reach it because of repulsion, and the wire thus has a negative potential considerably in excess of the region in which it is placed. If positive ions also were present, they would be drawn to the wire until its potential is the same as the surrounding region.

If two test electrodes are used, the field strength at various points through the discharge may be determined by measuring the potential difference between them and dividing by their distance apart. Except for regions close to the electrodes, where only one type of ion is present, this method gives reliable results. Because of the mechanical difficulty of moving a pair of test wires through a tube with fixed electrodes, it is more convenient to use a tube with fixed test wires *tt* and moveable electrodes as shown in Fig. 116. The anode *A* and cathode *C* are held at a fixed distance apart by means of a glass rod *d* with flexible leads connecting to the seals through the tube. A small lug of iron *I* is acted upon by a magnet so that the electrodes may be moved along the tube, placing the test wires at any desired part of the discharge.

158. Experiment 32. *Measurement of Field Strength in the Discharge through Air.*—Connect the apparatus as shown in Fig.

[1] GRAHAM, *Wied. Ann.*, vol. 64, 1898, p. 49.

116, using as a source of power either a battery of flash light cells or a motor generator set giving an E.M.F. of about 1,000 volts. Include a graphite resistance in series with the tube to prevent arcing when the conductivity is high. Measure the difference of potential between the test electrodes by means of an electrometer which has been checked against a standard voltmeter. Start the pump and note the character of the discharge from the highest pressure at which a current can be maintained to the best vacuum that the pump will give. An E.M.F. of 1,000 volts is not in general sufficient to start the discharge although it will maintain it, once it is going. To start it connect a small spark coil across the tube with an air gap in series to prevent shorting the generator or battery through the secondary of the coil. Determine the field strength at various points through the discharge for two pressures (*a*) the highest at which a uniform discharge can be maintained, (*b*) one at which the discharge has the characteristic appearance shown in Fig. 115. Measure the pressures by means of a McLeod gauge, and the voltage across the tube by an electrostatic voltmeter.

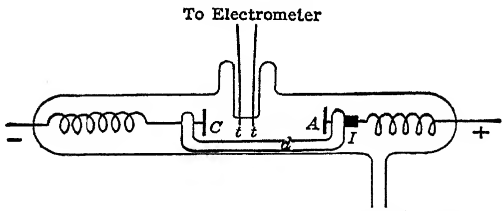

FIG. 116.—Tube for measuring potential gradients.

Report.—Indicate by sketches the character of the discharge for several different pressures. Plot field strength as a function of distance from the cathode for the two cases studied. Plot voltage as a function of distance from cathode. Obtain the latter from the area under the field strength—distance curve.

159. Cathode Rays.—It was pointed out above that when the pressure in a discharge tube has been reduced to a certain value, e.g., a hundredth of a millimeter of mercury, the character of the discharge is entirely changed from that represented by Fig. 115. The positive column shrinks back and disappears entirely and the Crooke's dark space occupies the entire volume of the tube. The glass now shows a bright fluorescence, green or blue, depend-

ing upon its composition. This fluorescence is due to bombardment by electrons shot out from the cathode or the region immediately in front of it. They travel in straight lines perpendicular to the cathode, and possess many interesting properties. For example, if they strike a piece of platinum foil, it may be heated to incandescence by their bombardment, or if they impinge upon substances such as willemite, calcium tungstate, barium platino-cyanide, etc., they cause them to fluoresce brilliantly. These streams of electrons are called cathode rays.

The fact that they possess a negative charge may be demonstrated by placing two parallel plates within the tube between which there exists a difference of potential. A stream of cathode rays passed between them will be deflected away from the negatively charged plate toward the positive. Again, if a magnetic field is introduced across the tube, the stream will be deflected at right angles both to their motion and to the field in the manner required by the ordinary rules of electrodynamic action for currents.

160. Velocity and Ratio of the Charge to the Mass of an Electron.[1]—The fact that an electron, when moving through a magnetic field, is acted upon by a force at right angles both to its motion and the direction of the field may be used to determine the ratio of the charge to the mass of an electron and the velocity with which it moves. Apparatus arranged for this purpose is shown in Fig. 117. A vacuum chamber C is constructed from a brass tube from which there projects a smaller tube A also of brass. The end of A is tapered and fitted to one end of a ground glass joint. The other end of the glass tube is closed and carries the cathode K. A piece of plate glass P, on the inner side of which has been placed a thin coating of fluorescent material such as calcium tungstate closes the vacuum chamber. The end of the smaller tube A contains a brass plug through which has been bored, with a jeweler's drill, a very fine hole.

When a suitable vaccum has been obtained, a discharge produced between A and K by a static machine M causes a stream of electrons to pass from K to A, the individual electrons of which move in straight lines normal to K. All but those lying in a very narrow beam, defined by the hole through A, are stopped,

[1] Townsend, Electricity in Gases, p. 453.
Crowther, Ions, Electrons and Ionizing Radiations, p. 92.
Duff, A Text Book of Physics, p. 492.

but those passing through, enter the chamber C and produce on P a bright fluorescent spot. If now the solenoid is energized, the magnetic field causes a deflection of the beam and the spot is moved a distance d perpendicular to the plane of the paper, (shown in the plane of the paper in Fig. 117).

If the magnetic field is uniform, the path of an electron is circular, since the force, in this case, is constant in magnitude, and is always at right angles to the motion. The magnitude of the force may be obtained as follows: Let l be the length of path of an electron through the magnetic field. When it has traversed

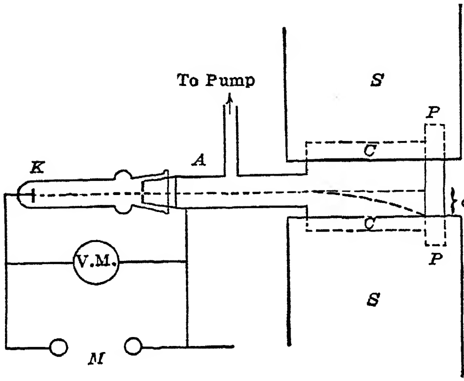

FIG. 117.—Apparatus for measuring $\frac{e}{m}$.

the distance l, a quantity of electricity e has been transported through this distance and may be replaced by a steady current of strength i defined by $i = \frac{e}{t}$, where t is the time required for the electron to travel the distance l. The theory of electrodynamics gives, for the force acting on a conductor of length l, carrying a current i, the expression

$$F = Hil = \frac{Hel}{t} = Hev \tag{1}$$

where v is the velocity with which the electron moves.

Since the electron moves in a circle, whose radius we will call R, the force given by eq. (1) must balance the centrifugal force. Accordingly, we have

$$Hev = \frac{mv^2}{R} \tag{2}$$

where m is the mass of the electron. The velocity v is acquired while the electron moves through the difference of potential E maintained between the anode and cathode by the static machine. Since there is no potential difference between A and P it travels this distance with constant velocity. The kinetic energy acquired in moving from K to A is equal to the loss of potential energy over this distance. From the law of conservation of energy and the definition of potential difference, we have

$$Ee = \tfrac{1}{2}mv^2 \tag{3}$$

Eliminating successively v and $\frac{e}{m}$ from eqs. (2) and (3) there results

$$\frac{e}{m} = \frac{2E}{R^2H^2} \text{ and } v = \frac{2E}{RH} \tag{4}$$

The radius of curvature R is obtained from the sagitta formula

$$R = \frac{l^2}{2d} \tag{5}$$

The magnetic field strength H is computed from the dimensions of the solenoid and the current through it by the formula

$$H = \frac{4\pi NI}{10L}$$

where N is the number of turns on the solenoid and L its length. The accelerating potential E is measured by an electrostatic voltmeter.

161. Experiment 33. *Measurement of* $\frac{e}{m}$ *and Velocity for an Electron in a Cathode Ray.*—Connect the apparatus as shown in Fig. 117. Start the static machine and pump the vacuum chamber until a green fluorescence is seen near the anode A. Should this color appear at K the leads to the static machine should be reversed. A bright spot will appear at P. Energize the solenoid and determine the current required for a suitable deflection d. In taking observations, reverse the solenoid current and measure $2d$. It will be found that by varying the vacuum, different voltages may be maintained across the discharge while the static machine is driven at a constant speed. With the two halves of the solenoid as close together as possible, take a series of observations using different accelerating voltages, and deflecting fields and determine $\frac{e}{m}$ and v. The fact that the parts of the solenoid must be separated to permit the entrance of the

discharge tube introduces a non-uniformity in the field. To determine this error, take a series of observations, keeping the accelerating potential and the solenoid current constant and increase the separation of the solenoid parts from the smallest amount up to 10 cms., and plot the apparent values of $\frac{e}{m}$ and v as a function of the separation. The intercept of this curve, when extrapolated to zero separation gives the correction to be applied to the results obtained above. Since the value of $\frac{e}{m}$ is usually given in electromagnetic units per gram, it is necessary to express E and H in eq. (4) in that system.

Report.—Plot the correction curve called for above and apply to average values of $\frac{e}{m}$ and v. Compute the velocity of an electron which has fallen through the following differences of potential using your value for $\frac{e}{m}$: 300, 3,000, 30,000 volts. Compute the time required for an electron to move from K to A for some one of the conditions actually used in this experiment. If the charge on an electron is 4.77×10^{-10} electrostatic units, compute the number of electrons passing per second across a plane in a wire through which a current of one ampere is flowing.

162. Radio-active Substances.[1]—If the region surrounding any radio-active substance such as uranium, radium, thorium, etc., is examined by appropriate means, it is found that these substances emit definite radiations which have very unusual properties. These radiations, for example, are able to darken a photographic plate, to convert an insulating gas into a conductor, and to cause a fluorescent screen to emit light. Moreover, they are different from ordinary light in that they are able to penetrate many substances usually regarded as opaque. It has been found that each radio-active substance is a definite chemical element and that its activity is due to a spontaneous decomposition or disintegration of its atoms. Furthermore, when certain of the rays are emitted, there is a definite reduction in the atomic weight of the substance, which naturally leads to the view that the atoms of these substances are made up of complex systems

[1] Crowther, Ions Electrons, and Ionizing Radiations, chap. XI.

McClung, Conduction of Electricity through Gases and Radioactivity. Part II.

Duff, A Text Book of Physics, p. 502.

which have the same intrinsic character and differ from one another only in their order of arrangement or degree of complexity. Three distinct types of radiation have been found which are designated as α, β, and γ rays.

163. The Alpha Rays.—These rays are distinguished from the others by the fact that they are easily absorbed on passing through gases or thin sheets of metal and that their action on a photographic plate is weak. On the other hand, they are very effective as a means for ionizing a gas, and they cause fluorescent substances to emit light. If a screen upon which they are acting is examined by a microscope, it is found that the illumination is not uniform but is made up of a large number of separate flashes as though the screen were under bombardment. In fact it has been found that α rays are discrete particles shot out from radio-active substances and it is possible by suitable experimental arrangements, to photograph their zig-zag courses as they make their way through a gas, abruptly deflected by some of the gas molecules, and stopped by others.

If a beam of α particles is shot at right angles to an electric or a magnetic field, the path is curved in much the same manner as the cathode ray stream described above, except that the deflection is much smaller in magnitude due to their larger mass and is in the opposite direction, indicating that they are positively charged. By making use of electric and magnetic deflections, the value $\frac{e}{m}$ of the ratio of the charge to their mass and the velocity with which they are emitted, have been measured. The results show that $\frac{e}{m}$ is the same for all α particles, no matter what their source and is equal to 4,823 electromagnetic units per gram, and that the velocities range from 1.5×10^9 to 2.2×10^9 cms. per sec.

The ratio of the charge to the mass for the hydrogen ion in electrolysis is twice that for the α particle, and at first sight it might be supposed that the latter is a hydrogen molecule consisting of two atoms. However, it has been found that the charge carried by the α particle is twice that of the hydrogen ion, and hence its mass must be four times that of the hydrogen atom. Since the particle is atomic in size and is of the same order of magnitude as the atom of helium whose atomic weight is 4.00, the most natural assumption is that it is an atom of

helium with twice the electric charge of the hydrogen ion. This hypothesis is supported by the fact that both chemical and spectroscopic analyses show conclusively that helium is always present where radio-active transformations are taking place.

164. The Beta Rays.—The β rays are distinguished from α rays in several important respects. In the first place, they have a far greater penetrating power. While the α rays are completely stopped by a sheet of aluminum foil $\frac{1}{10}$ mm. in thickness, β rays still produce noticeable effects after passing through sheets 100 times this thickness. Again, they are much more easily deflected by a magnetic field. The deflection of the α rays is appreciable only in the largest fields available, and even then special methods have to be employed. The β particles, on the other hand, travel in circles of large curvature when moving at right angles to fields of ordinary magnitudes. The direction of the deflection shows that they carry a negative charge; and all the evidence indicates that they are identical with the cathode rays of the ordinary discharge tube, i.e., electrons.

By subjecting β rays to the deflecting action of electric and magnetic fields combined, the values of $\frac{e}{m}$ and the velocities with which they are emitted may be measured. It has been found that while the former is the same as for the cathode-ray particles, the velocities of emission are considerably higher than those observed in discharge tubes, ranging from 6×10^9 to 2.8×10^{10} cms. per seond. The latter is very close to the velocity of light, 3×10^{10} cms. per second.

A careful study has been made by Kaufmann of the value $\frac{e}{m}$ for the particles as a function of velocity, and it was found that $\frac{e}{m}$ is not constant, but decreases as the speed increases. This can be explained only by assuming that e decreases or that m increases as the velocity becomes larger. The evidence furnished by other lines of study indicates that the charge of the electron is one of the fixed constants of nature, and therefore it is concluded that the mass of the electron depends upon its velocity. Theoretical considerations have shown that the apparent mass of an electron is due wholly, or in part, to the motion of its electric charge. In fact, for a number of years, the view was held that the mass of the electron is entirely electromagnetic in character, but some very recent work indicates that this can not be the case entirely.

165. The Gamma Rays.—The nature of the γ rays is very different from that of the α and β rays. They are distinguished by the fact that they possess very much greater power of penetration. In fact they may easily be detected after passing through several cms. of iron. Though subjected to the most powerful electric and magnetic fields available, they show no deflection, and can not therefore carry an electric charge. They cause a fluorescent screen to emit light, and affect a photographic plate. When passed through gases they produce ionization, and, in fact, are usually detected by this action.

Searching investigations have shown that they are similar in character to X-rays, that is, electromagnetic waves of very short wave length. The similarity of the relation of β rays to cathode rays and γ rays to X-rays is very close. When the target of an X-ray tube is struck by a rapidly moving electron, the electronic orbits of one of the atoms of the former undergoes some sort of rearrangement; that is, they change over from one stable configuration to another possessing a different amount of potential energy, and a train of X-rays is emitted. The emission of the X-ray occurs as the result of suddenly stopping a high speed electron. Similarly, when a radio-active substance emits a β particle, sending it forth with a velocity comparable to that of light, a rearrangement of the electronic orbits also occurs, which is accompanied by the emission of the γ ray. The γ rays thus accompany the rapid acceleration of electrons. The fact that γ rays are always present when β rays are emitted supports this view. Measurements have shown that the wave length of the γ rays is somewhat shorter than that of the most penetrating X-rays.

166. Radio-active Transformations.—Careful investigations of the phenomena accompanying the emission of the rays just described, show that radio-active substances are distinguished from ordinary ones in that they are constantly undergoing changes of character, never observed in ordinary materials. Each substance is entirely distinct from the other, and has its own characteristic physical and chemical properties. However, instead of enduring indefinitely as is the case with ordinary elements, such as copper, iron, gold, etc., each radio-active substance has a definite, measurable period of existence, after which it disintegrates and becomes a new chemical substance, and it is during these processes of transformation, that the emission of rays occurs.

All molecules are made up of atoms which consist of positive nuclei with electrons rotating about them in closed orbits. The electrons are held in their orbits by the electric attractions existing between them and the nucleus while the atoms are held together by the electric forces between their parts, or the magnetic forces due to the circulating electrons. This complicated structure becomes unstable for some reason or another, and an α or a β particle or both is emitted. After a rearrangement of the remaining particles, a new state of stable equilibrium ensues, giving a new substance of different physical and chemical properties. As an illustration, take the substance radium. Although the individual atoms do not have the same periods of existence, the average life of an atom is 2,000 years. When a radium atom disintegrates, it emits an α particle, and the residue is called radium emanation. The emanation persists for an average period of 3.75 days when it gives off another α particle and becomes radium A. In this form it lasts for 3 minutes, then again emits an α particle and becomes radium B. This state persists for 26 minutes when it gives off a β particle accompanied by a γ ray and becomes radium C, and so on. The entire series has been carefully worked out, starting with uranium, going through ionium and the various phases of radium, and thorium to those of actinium. The duration of the different phases ranges from a few minutes to 10^{10} years. Some of the transformations are apparently not accompanied by the emission of any rays. These transformations are explained by supposing that the ray is present but possesses such a low velocity as to be unable to ionize a gas and is therefore not detected.

It is important to note that each time an α particle is emitted, the atomic weight decreases by 4, i.e., the atomic weight of helium. Furthermore the last radio-active product, radium F, or polonium, transforms into an element which has the atomic number of lead and is probably one of the isotopes of lead. It is easy to conjecture that each of the chemical elements, as we know them, has been derived from one higher in the scale of atomic weights by the emission of one or more α particles, and that transformations are going on continuously but at a rate so slow as to escape detection by methods at present available.

167. Experiment 34. *Ionization by Alpha Particles from Polonium.*—The apparatus for this experiment is shown in Fig. 118. It consists of an ionization chamber made entirely of metal.

The radioactive substance is coated on plate A which may be moved up or down. An insulated plate D is connected to an electroscope E mounted in another chamber and connected with the ionization chamber by a removable brass tube. The electroscope is charged by means of a battery B of small dry cells by pressing down the wire W which must be insulated from the container. When the rays from the radioactive substance pass up through the metal gauze G they ionize the air between G and D. Either electrons or positive ions, depending upon the sign

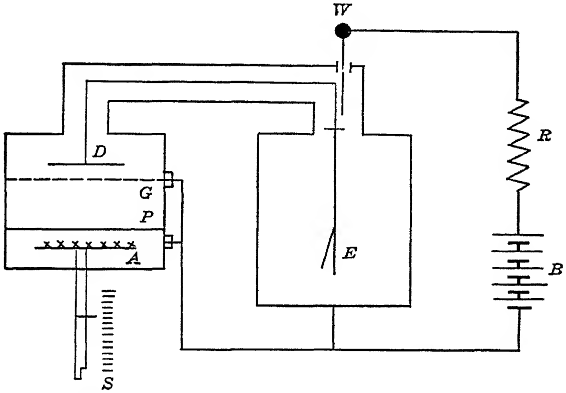

FIG. 118.—Apparatus for ionization studies.

of the charge on D and E, are drawn toward D and neutralize this charge. The deflections of the electroscope are read by means of a long-focus microscope provided with an eyepiece having a graduated scale. The time required for the gold leaf of the electroscope to fall through one division is inversely proportional to the ionization current.

It is necessary first to determine the rate of discharge of the ionization chamber due to leakage alone. To do this, the plate A is pulled as far down as possible. The distance AG is then greater than the range of the α particles from polonium so that no ionization is produced in the portion of the chamber above the gauze. Charge the electroscope by connecting it for an instant to the battery by means of W and note the time required for the gold leaf to fall one division. Take several readings and average. The reciprocal of this time is a measure of the leakage current across the insulation. Next, move A to its top position, i.e., to its position nearest the gauze, and in the same manner measure

the time of leakage. Repeat, moving A down in steps of 3 mm., until the time of leakage becomes comparable with the value found for the lowest position of A.

Report.—Plot inverse time of leakage as ordinates against the separation of A and G as abscissas, correcting the ordinates for the leakage across the insulation. Extrapolate the steepest part of the curve to get the average range of the particles. Explain the form of your curve.

167A. Motion of a Charged Sphere through Air.[1]—When a particle moves through a viscous medium under the action of a force it is subject to a frictional drag which is proportional to the velocity. The particle will be accelerated until it reaches a velocity where the frictional force equals the accelerating force when no further acceleration takes place and the particle moves with constant velocity.

Stokes has shown that for spherical particles the frictional drag is given by the expression.

$$F = 6\pi\eta a V$$

where η is the coefficient of viscosity of the medium, a is the radius of the drop, and V its velocity.

Suppose a small drop of oil is moving through air under the action of gravity. We see from the foregoing that the terminal velocity will be given by

$$mg = 6\pi\eta a V_1$$

If, now, the drop carries a charge, e_η, and is moving under the action of an electric force X parallel to the gravitational force but oppositely directed, the terminal velocity will be given by

$$Xe_\eta - mg = 6\pi\eta a V_2 \qquad (2)$$

Dividing (2) by (1) we get

$$\frac{Xe_\eta - mg}{mg} = \frac{V_2}{V_1}$$

Thus if we know m, V_2, V_1, and X, e_n may be determined. This fact was used by Millikan in his determination of the electronic charge.

X, V_2, and V_1 may be readily determined and m may be found from Stokes' law, eq. (1). From this we get

$$m = \frac{9\pi\sqrt{2}}{\sqrt{\rho}}\left(\frac{\eta V_1}{g}\right)^{3/2}$$

[1] HARNWELL and LIVINGOOD, "Experimental Atomic Physics." MILLIKAN, "The Electron."

where ρ is the density of the drop, and buoyancy due to the air is neglected.

For liquid drops of the size given by an ordinary atomizer the terminal velocity is reached very quickly.

The experimental arrangement is illustrated in Fig. 118*a*. CC' are two parallel condenser plates, the upper one of which is perforated so that the oil spray from an atomizer may penetrate to the space between the two plates. L is a source of light (a flash-light bulb is sufficient) for illuminating the drops and thus rendering them visible through the microscope M.

The time for the drop under observation to fall or rise a given distance, as determined by a calibrated scale in the microscope eyepiece, can be measured with a stop watch. Then knowing the field (V/d) between the condenser plates (where V is the

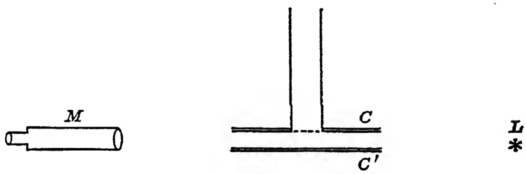

FIG. 118*a*.

voltage and d is the distance between the plates), the density of the drop and the viscosity of the air (.0001823), the charge on the drop may be determined.

If the drops are small, i.e., if they fall slowly through the air under the gravitational action, a correction to Stokes' law is necessary. It is convenient in order to avoid this correction to use relatively large drops, i.e., drops whose velocities lie between .002 and .05 cm. per second.

The ionization necessary for the charging of the drop may conveniently be produced by placing a small amount of some gamma ray source near the apparatus.

In general the charge on a drop will be some multiple of the electronic charge so that it is necessary to measure the charge on a number of drops and to find the common factor. This will then be the electronic charge.

167B. Experiment 34A. *Determination of the Electronic Charge.* The scale in the microscope eyepiece is to be calibrated by sighting on a millimeter scale etched on glass which is furnished with the instrument. The windows of the condenser chamber must

be clean and may be slipped out if necessary to wipe them off. The beam of light should enter the chamber at an angle of about 135° to the line of sight. The microscope is focused on a pin or piece of wire inserted through the central hole in the upper condenser plate. When properly adjusted, the background appears fairly dark and the drops are bright points of light. The oil is sprayed gently diagonally across the top of the tube which fits into the upper plate. The cork should be replaced to prevent air currents. The drops should now be visible as they fall through the field of view of the microscope. Only as much oil as is absolutely necessary should be sprayed into the chamber in order to prevent clouding of the windows.

When the rocker switch on the bases is in a vertical position the condenser plates are connected to each other and the field between the plates is zero. The drops then move under the action of gravity alone. The other positions apply the potential to the plates either direct or reversed. With drops falling through the field of view of the microscope, this switch is rocked back and forth until a drop reverses its motion under the action of the field. The upward and downward motion of this "captured" drop is then studied.

Report.—Determine the total charge on at least six drops (or different charges on the same drop) and from this compute the electronic charge.

CHAPTER XIV

PHOTOMETER[1] AND OPTICAL PYROMETER

168. Intensity of Radiation.—The brightness of light, as estimated by the eye, is not capable of precise measurement, since it depends to a large extent upon the color of the light and the sensitiveness of the eye which receives it. Accordingly, the only consistent way in which intensity may be specified is in terms of energy. Proceeding on this basis, the intensity of waves, whether they are those of sound, light or of any other type, is measured by the amount of energy passing per second through a square centimeter of area at right angles to the direction of propagation. If there is no loss in the medium, and if the medium contributes nothing to the intensity, the same quantity of energy will persist in a given wave no matter how far it travels, or how the dimensions and form of the wave front may change as it advances.

The variation of intensity with distance from the source depends upon the shape of the wave front, or what amounts to the same thing, the number of dimensions in which the wave spreads out. For example, if the wave front is plane, as in the case of a sound wave travelling along a speaking tube, or the beam from a searchlight, the wave front maintains a constant area, and the intensity is independent of the distance from the source. Again, if a pebble is dropped in the lake, waves travel outward in circles and are propagated in two dimensions. In this case the energy remains constant in a circle which increases as the distance from the center and the intensity varies inversely as the distance from the source. In the case of spherical

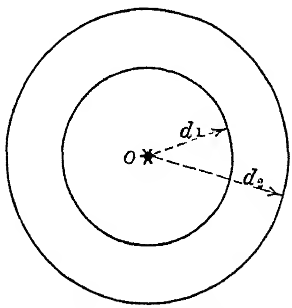

FIG. 119.—Propagation of spherical waves.

[1] DUFF, Text Book of Physics, arts. 259, 637–639, 724.
KARAPETOFF, Experimental Electrical Engineering, arts. 205–211.
NUTTING, Outlines of Applied Optics, p. 169.

waves with which we are particularly concerned here, the energy emitted per vibration of the source is confined within a spherical shell whose thickness is that of one wave length, and this remains constant as the wave advances. Let O, Fig. 119, be a source from which waves are sent out in all directions. Let S be the strength of the source, i.e., the amount of energy emitted per second. Also let d_1 and d_2 be the radii of a given wave at two different distances from the source and let I_1 and I_2 be the corresponding intensities. Then

$$S = 4\pi d_1^2 I_1 = 4\pi d_2^2 I_2 \tag{1}$$

whence

$$\frac{I_1}{I_2} = \frac{d_2^2}{d_1^2} \tag{2}$$

Thus the intensity varies inversely as the square of the distance from the source.

169. The Photometer.—An instrument for the comparison of two sources of light is called a photometer. While the eye is unable to estimate absolute intensities at all accurately, it is, nevertheless, quite sensitive to differences in illumination.

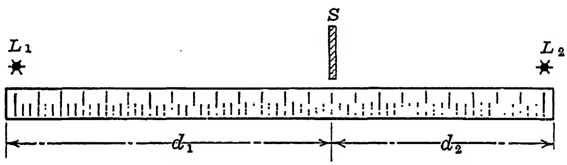

FIG. 120.—Principle of the photometer.

Accordingly, if light from two different sources is allowed to fall upon a screen in such a way that the areas of the separate illuminations are adjacent, equality in the two intensities may be determined by the disappearance of the line of demarkation between them. An instrument for this purpose may be arranged as shown in Fig. 120 by mounting two lamps L_1 and L_2, which are to be compared, at the ends of a bench provided with a scale along which runs a carriage supporting a screen of white paper. The central portion of this screen is impregnated with paraffine which renders it semitransparent. This spot appears darker than its surroundings if viewed by reflected light, but it is brighter in transmitted light. If, however, the intensity of illumination is the same on both sides, the spot disappears since the amounts transmitted in the two directions are equal.

If S_1 and S_2 are the strengths of the two sources and d_1 and d_2 their respective distances from the screen, then by eq. (1) the ilumination on each side of the screen is given by

$$I = \frac{S_1}{4\pi d_1^2} = \frac{S_2}{4\pi d_2^2} \tag{3}$$

$$\frac{S_1}{S_2} = \frac{d_1^2}{d_2^2} \tag{4}$$

If one of the sources, e.g., S_2 is a standard lamp, S_1 may be computed.

170. The Lummer-Brodhun Photometer.—A comparator considerably more sensitive than the grease spot screen just described has been developed by Lummer and Brodhun. The special

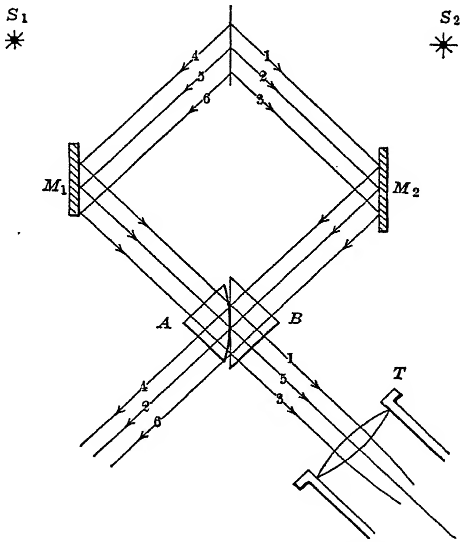

FIG. 121.—Lummer-Brodhun photometer.

feature of this instrument is the optical device for simultaneously viewing the two sides of the comparison screen W, as shown in Fig. 121. Light from each side is reflected by two mirrors or prisms M_1 and M_2 so as to enter the optical system AB. This consists of two totally internally reflecting prisms placed back to back. The reflecting surface of one is plane, while that of the other is spherical with a small portion ground flat. The flat surface of the latter is placed in optical contact with the former.

Light entering either of these prisms and striking the contact surface will be transmitted, but light striking any portion of the reflecting surfaces backed by air will be totally internally reflected. Light emerging from the prism B consists of two parts, that from the contact portion of the two prisms and that from the surrounding area. The former comes entirely from the left-hand side of W while the latter is from the right-hand side. If a telescope is placed at T and focused on the contact area of the two prisms, the central portion appears brighter or darker than the surroundings according as the illumination of the left- or the right-hand side of W is more intense, but the entire field appears uniformly illuminated when a balance is secured.

A convenient form of laboratory instrument is one in which a single socket, to receive in succession the unknown and standard lamps, is mounted at a fixed distance from the comparison box. On the other side is a movable socket containing a small six volt lamp for comparison purposes. The distance of this lamp from the screen is read by an index registering on a fixed scale. A slow motion device is also provided. The process consists then in placing the unknown lamp in the fixed socket and obtaining a balance by moving the comparison lamp to or from the screen until the line of demarkation between the outer and central positions of the field of the telescope disappears. The lamp is then replaced by the standard and a balance again obtained. The screen should be reversed and readings taken in each position and averaged to eliminate differences in reflecting power of the two sides. The equation for computing the strength of the unknown lamp may be derived as follows:

Let S, U, and C be the candle powers of the standard, the unknown, and comparison lamps, respectively; let d_s and d_u be the distances of the comparison lamp from the screen when balanced against the standard and unknown, and let D be the fixed distance of both standard and unknown from the screen. Then, for the two balances, the following equations hold:

$$\frac{U}{C} = \frac{D^2}{d_u^2} \qquad \frac{S}{C} = \frac{D^2}{d_S^2}$$

Dividing one equation by the other, we have

$$\frac{U}{S} = \frac{d_s^2}{d_u^2} \tag{5}$$

Care must be taken to maintain the same voltage on the comparison lamp throughout the test.

171. Experiment 35. *Study of Incandescent Lamps.*—The purpose of this experiment is to determine, as a function of the voltage upon which they are operated, the candlepower, wattage consumption, watts per candlepower, and resistance of four lamps differing as widely as possible in design. Each lamp, including the comparison lamp, should be provided with a voltmeter and a control rheostat. An ammeter should be placed in series with the unknown. Use five different voltages between 90 and 130. Do not operate the lamps at the higher voltages longer than is necessary for making the observations. The standard lamp should be operated only at the voltage for which it is rated. Make several settings for each observation using the screen in both the direct and reversed positions.

Report.—Describe the Lummer-Brodhun photometer and plot the four curves indicated for each lamp. Why does a tungsten lamp reach full brilliancy more quickly after closing the switch than a carbon? Why does the gas-filled lamp have a higher efficiency than a vacuum lamp?

THE OPTICAL PYROMETER[1]

172. General Principles.—It is a matter of common experience that when a body is heated to a high temperature it emits light and also that the intensity of this emitted radiation varies rapidly with the temperature of the source. For example, a small change in the voltage across an incandescent lamp produces a relatively large change in the brightness of the filament. Measurements show that a body at 1,500° C. emits more than one hundred times as much as it does at 1,000° C., and if the temperature is raised to 2,000° C., the radiation is increased more than two thousand fold. This fact is often made use of in the measurement of temperatures, and pyrometers operating on this principle have the marked advantage that it is not necessary to heat any part of the measuring apparatus to the temperature of the body being studied. This is particularly important for work above 1,600° C., for there is no substance which retains its temperature measuring properties uniform when subjected to such extreme heats. Again, the products of combustion in furnaces contaminate any pyrometric material introduced, thus necessitating frequent recalibrations.

[1] LeChatelier and Burgess, Measurement of High Temperatures, pp. 237–243, 291–303, 325–327, 336–337.

Griffiths, Methods of Measuring Temperature, pp. 113–118.

The radiation method of measuring temperatures, however, is complicated by the fact that incandescent bodies differ materially as regards both the intensity and quality of the light which they emit. For example, the radiation from iron or carbon is much greater than that from such substances as magnesia or polished platinum at the same temperature. If a pyrometer were calibrated by measuring the radiation from one substance and then used to measure the temperature of another possessing different radiating properties large errors would result in many cases.

This difference in radiating properties has led to the use of "black bodies" as standard radiators and absorbers. A black body is defined as one which absorbs all the radiation falling upon it, and it therefore neither reflects nor transmits any radiation. It also has the property, when heated, of emitting radiation whose intensity is a function of temperature only and depends in no way upon the physical constants of the material of which it is made. Further, the intensity of the radiation from a black body at a given temperature is greater than that from any other body at the same temperature.

173. Black Body Furnace.—Experimentally, a black body is very closely approximated by a hollow opaque inclosure with a small opening. If the internal area of the inclosure is large compared to the opening, radiation falling upon it enters the inclosure and is reflected diffusely back and forth so many times, that it is practically all absorbed before any can emerge. Again, if the walls are heated uniformly to any temperature, the radiation emerging from the opening has been reflected back and forth so many times that it no longer has properties characteristic of the material of the walls. Such a body is at the same time a perfect absorber and a perfect emitter. The radiation from a crack or other small opening in an ordinary furnace is nearly black body radiation, so also is that from the inside of a narrow wedge formed by folding a thin metallic ribbon into a very flat *V*.

A black body, satisfactory for experimental purposes, is made by winding a porcelain tube with thin platinum foil through which a heating current may be passed. The center of the tube is closed by a porcelain disk and between this and the end, through which observations are made, is arranged a series of diaphragms, also of porcelain, whose apertures increase in diameter successively toward the end. These minimize the disturbing effects of air currents and increase the number of internal reflec-

tions which the radiation must make before it emerges. To protect the internal tube from external disturbances and reduce the heat losses to a minimum, it is surrounded by another tube upon which is wound a second heating coil of some alloy such as nichrome or therlo. Outside of this is a series of several additional tubes with air spaces between them, the outer one usually being surrounded by powdered magnesia. By properly adjusting the heating currents through the two coils, any desired temperature up to 1,600° C. may be maintained with a high degree of constancy. The temperature of the black body is usually measured by a platinum, platinum-rhodium thermocouple, the junction of which is supported by two small holes through the central disk, with the insulated leads passing out through the rear of the furnace.

174. Distribution of Energy in the Spectrum.—If one measures the total energy emitted by a black body, he finds that it increases rapidly as the temperature is raised. The law connecting black body radiation with temperature was first stated by Stefan and later deduced theoretically by Boltzmann. It is

$$E = ST^4 \tag{6}$$

where E is the total energy radiated, T the absolute temperature, and S, a constant which is approximately 5.6×10^{-5}, ergs per square centimeter per second. Although this law is rigidly true only for a black body it is found to hold approximately for most surfaces, the constant S being different for each.

If the radiation from a black body is separated out into a spectrum and the energy associated with each wave length is measured, it is found that not only is there a continuous change in the amount of energy as we go from one wave length to another, but also that the distribution of energy among the wave lengths changes as we vary the temperature. Figure 122 gives the distribution of energy among the wave lengths for a series of temperatures. It will be noted that as the temperature is raised, the energy in each wave length increases but not in the same proportion. Also that the wave length containing the maximum energy decreases as the temperature is raised. This is in accord with the common observation that, starting with low temperatures, a body appears at first dull red, then yellowish or cherry red, and finally becomes "white hot" as extreme temperatures are reached. Wien has shown that the wave length for maximum

energy and the absolute temperature are connected by the simple law

$$\lambda_{\max} T = \text{const.} \tag{7}$$

He has also shown that the distribution of the energy among the wave lengths at a given temperature, as illustrated by Fig. 129, follows very closely the law

$$E_\lambda = c_1 \lambda^{-5} e^{-\frac{c_2}{\lambda T}} \tag{8}$$

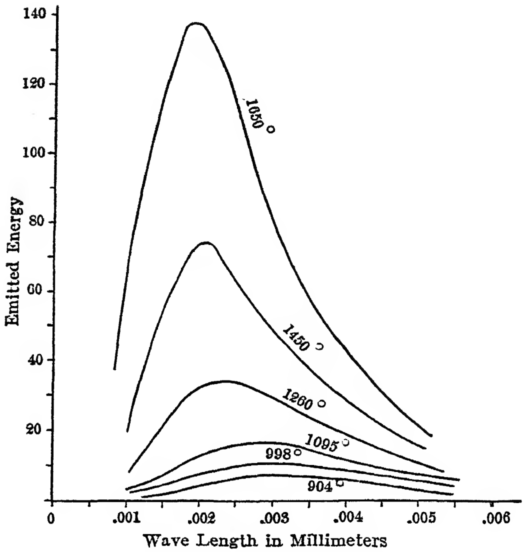

FIG. 122.—Energy distribution for a black body.

where E_λ is the energy in the wave length interval λ to $\lambda + d\lambda$; e is the base of the Naperian logarithms; T the absolute temperature, and c_1 and c_2 are constants. For other radiating surfaces, it is found that E_λ follows very closely the above law but different constants must be used.

175. Application to Pyrometry.—It is obvious that any of the three equations just given might be used to measure temperatures. It is found, however, that eq. (8) is most suitable, and when it is applied, only one wave length is used, or at least only those lying within a very restricted range. This equation lends itself more easily to calculation if it is put in the form:

$$\log_e E_\lambda = k - \frac{c_2}{\lambda T} \tag{9}$$

where

$$k = \log_e c_1 - 5 \log \lambda.$$

Let E_1 and E_2 be the energies for a particular wave length radiated at the temperatures T_1 and T_2, respectively. Substituting these values in eq. (9) and subtracting, we have

$$\log_e \frac{E_1}{E_2} = \frac{c_2}{\lambda}\left(\frac{1}{T_2} - \frac{1}{T_1}\right) \qquad (10)$$

If T_2 is a standard temperature and T_1 an unknown, then by measuring E_1 and E_2 or their ratio, by appropriate means, T_1 may be computed. Solving eq. (10) for T_1 and using common logarithms,

$$T_1 = \frac{c_2}{\lambda} \quad \frac{1}{\frac{c_2}{\lambda T_2} + 2.303 \log_{10} \frac{E_2}{E_1}} \qquad (11)$$

176. The Optical Pyrometer.—One of the most convenient forms of the optical pyrometer is that devised by Holborn and

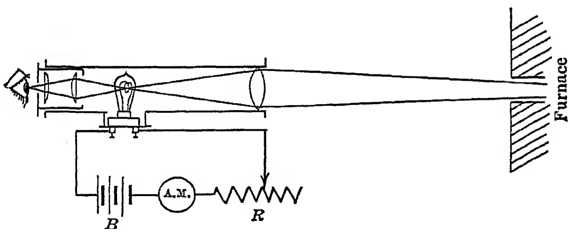

FIG. 123.—Holborn and Kurlbaum optical pyrometer.

Kurlbaum. It consists of a telescope in the focal plane of which is mounted a small six volt lamp with either a carbon or tungsten filament, as shown in Fig. 123. When the telescope is focused on the furnace and the filament is lighted, there is seen, on looking into it, a field of uniform illumination with a fine line extending across it. If the filament is hotter than the furnace, it appears as a bright line across a dark background; but if the furnace is hotter, there is seen a dark line across a bright background. If filament and furnace are at the same temperature, the line disappears and the field is uniform throughout. The eye is very sensitive to differences of brightness and a difference of two degrees between furnace and filament may easily be detected. Current for the filament is furnished by a storage battery, controlled by a

rheostat and measured by an ammeter. The indications of the pyrometer are thus in terms of the filament current. If the furnace is held in turn at a series of known temperatures and the filament currents for balance obtained, a calibration curve may be plotted showing temperature as a function of current.

A number of improvements in the original form of the Holborn-Kurlbaum pyrometer have been made by Mendenhall.[1] One of them is a method by which such an instrument may be calibrated over a wide range using only one standard temperature. This is accomplished by holding the temperature of the furnace constant and rotating between it and the pyrometer a sectored disk which allows only a known fraction of the energy to enter the telescope. This is equivalent to reducing the temperature of the furnace. Suppose the fraction of the energy transmitted is R. Then $E_1 = RE_2$. Substituting this value in eq. (11), we have, for the apparent temperature of the furnace,

$$T_1 = \frac{c_2}{\lambda} \frac{1}{\frac{c_2}{\lambda T_2} + 2.303 \log_{10}\frac{1}{R}} \tag{12}$$

By using a series of sectors, for example with R equal to $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., a series of apparent temperatures are obtained, and the filament temperatures corresponding to each may be determined. This gives a calibration for the instrument for ranges below the standard temperature actually maintained in the furnace. The necessary narrow wave length band is secured by mounting behind the eyepiece a disk of red glass of special quality. The instrument also may be used to measure temperatures above that of the standard by using the sectored disk when taking observations on the unknown temperature, thus reducing it to an apparent lower temperature within the calibration range just determined. For example, if an unknown temperature is observed through a sector of transmission ratio R and is found to be the same as the standard temperature T_2 then the unknown temperature is obtained from eq. (11) by putting $E_2 = RE_1$ which gives

$$T_1 = \frac{c_2}{\lambda} \frac{1}{\frac{c_2}{\lambda T_2} + 2.303 \log_{10}R} \tag{13}$$

In a similar manner a calibration curve may be computed for a given sector extending the range of the instrument to any desired

[1] Mendenhall, *Phys. Rev.*, vol. 35, 1910, p. 74.

value. For this purpose, it is only necessary to substitute for T_2 in eq. (13) the value of temperature corresponding to each particular current read off from the original calibration curve. These computed values of T_1 plotted against the corresponding values of filament current give the calibration curve for the instrument when used with the sectors to measure unknown temperatures.

It should be borne in mind that the Wien radiation law upon which this method is based holds only for black body radiation, and that the method of calibration just described makes use of a black body as a source of radiation. If it should be used to determine the temperature of some other body such as a heated filament or strip of metal not within an enclosure, its indications will be the temperature of a black body which would emit the same amount of energy at the particular wave length used in the calibration. Since no other body emits more energy at any wave length than a black body at the same temperature, and most substances emit less than a black body, the reading of the optical pyrometer will, in general, be too low. The reading obtained is called the "black body temperature." Mendenhall and Forsythe[1] have made an extended study of the differences between the "black body" and "true" temperatures of a great many substances with the result that the optical pyrometer may now be very generally used to determine actual temperatures. A few of their values for carbon and tungsten are given below:

Black body temperature	Corresponding true temperature	
Degrees Centigrade	Tungsten, Degrees C.	Carbon, Degrees C.
1,000	1,068	1,012
1,200	1,273	1,222
1,400	1,486	1,430
1,600	1,700	1,638
1,800	1,910	1,847
2,000	2,126	2,056
2,200	2,345	
2,400	2,565	
2,600	2,783	
2,700	2,890	

[1] MENDENHALL and FORSYTHE, *Astrophysical Jour.*, vol. 37, 1913, p. 389.

177. Experiment 36.—Connect the apparatus as shown in Fig. 123. Ascertain the heating currents to be used through the two windings of the furnace and take care that they are never exceeded, particularly through the inner platinum winding. Find out, also, the maximum current allowable for the filament of the pyrometer lamp Fill the vessel containing the cold junction of the thermocouple with cracked ice to maintain it at 0° C. Measure the E.M.F. of the thermocouple with a low resistance potentiometer, special instructions for which are given in chap. IV. A calibration curve is furnished with the thermocouple. Focus the eyepiece of the telescope on the lamp filament and as soon a the furnace is warm enough to permit it, focus the telescope so that the inner circle of the furnace is distinctly seen. As the furnace heats up, determine its temperature with the thermocouple and balance the pyrometer every few minutes.

When a temperature of 1,200° C. has been reached, reduce the heating current through both windings and allow the temperature to rise slowly to about 1,300° C. and then hold the furnace constant at this value. When holding the temperature constant, it is best to leave the potentiometer setting fixed and keep the galvanometer balanced by adjusting the heating current rheostat. When a steady state has been secured, make several settings of the pyrometer. Then introduce the ½ sector and with the motor running, again make several settings on this apparent temperature. Repeat, using the ¼, $1/10$, $1/30$, $1/60$, and $1/120$ sectors. Check the settings with no sector between each replacement to insure constancy of conditions. Two observers are required for this experiment, one to manipulate the pyrometer, and one to hold the furnace temperature constant. Measure the temperature of the filaments of several incandescent lamps of different types and candle power using such a sector that the current through the pyrometer lamp lies within the range covered by the calibration.

Report.—Compute the effective temperatures below the standard temperature secured by the various sectors by use of eq. (12). Use for c_2 the value 14,350 and for the wave length 0.658. The standard temperature is that in degrees absolute at which the furnace was held constant and is obtained from the calibration curve for the thermocouple. The computed values of T_1 are also in degrees absolute. Plot temperatures below that of the furnace.

Plot calibration curves for values above that of the furnace for the $\frac{1}{10}$, $\frac{1}{30}$, and $\frac{1}{120}$ sectors, by use of eq. 13. To do this, read from the first curve the values of T for a series of values of filament currents. Substitute these values of T_2 in eq. 13 using for R the appropriate ratio. These values of T, when plotted against the currents, give the calibration curve for a given sector for the high range.

Read from these curves the black body temperatures of the lamps measured and by use of the tables given above, determine their true temperatures in degrees centigrade.

If the temperature of the sun is about 6,000° C., find the size of sector opening necessary to measure it on the instrument used.

CHAPTER XV

FUNDAMENTAL RADIO FREQUENCY MEASUREMENTS

178. Introduction.—Classifying the frequencies used in electrical work it is customary to divide them into three groups known respectively as audio, carrier, and radio frequencies. The boundaries of these regions are not sharply defined and they often overlap. While the human ear responds to frequencies up to 30,000 cycles per second, nevertheless telephone conversations can be carried on using frequencies up to 2,000 cycles per second, and reasonably good musical reproduction can be secured by utilizing frequencies up to 5,000 cycles per second. Carrier frequencies, i.e., those used for so-called carrier currents over wire telephone lines range from 15,000 to 50,000 cycles per second, while in radio communication the frequencies range from 20,000 to several million cycles per second.

Measurements carried out in the radio-frequency range, because of the small magnitudes of inductance and capacitance employed, require an entirely different technique from that used in low-frequency or direct-current measurements. The circuits are much simpler than those of the complicated bridges discussed above and electrical resonance plays an important role. Currents are, in general, measured by instruments of the thermocouple type, voltages by specially constructed vacuum-tube voltmeters, while the circuits themselves are energized by vacuum tube driven oscillators.

179. The Frequency Meter.—This is one of the most generally useful instruments in a radio laboratory, for it measures not only the frequency of any circuit, from which the wave length may be computed, but also, when proper auxiliary standards are available, inductances and capacitances as well. It consists primarily of a simple series resonance circuit including an inductance, capacitance, and some device for indicating the presence of oscillations. The condenser is usually of the smoothly variable air dielectric type, while the inductance is variable by fixed steps and consists either of a tapped coil or of several coils with interchangeable mountings. The indicator is usually a sensitive thermo-galvanom-

eter joined in series with the circuit although a voltage indicator is sometimes connected across the condenser instead. One commercial form is shown in Fig. 124.

To measure the frequency of any oscillatory circuit it is merely necessary to bring the frequency meter in inductive relation to it and adjust its tuning element until resonance occurs as shown by a maximum reading of the indicating instrument. If the values of the inductance and capacitance are known, the frequency is given by the equation

$$f = \frac{1}{2\pi\sqrt{LC}} \tag{1}$$

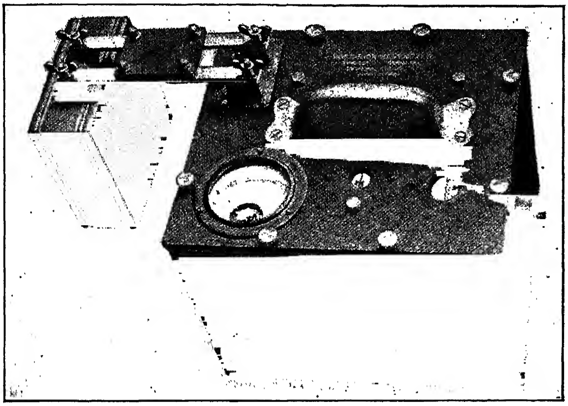

FIG. 124.—General radio frequency meter.

where L and C are expressed in fundamental units, i.e., henries and farads respectively. Usually the instrument is provided with a calibration curve from which frequencies or wave lengths may be read directly from condenser settings. For multi-range instruments the inductances are so chosen that the calibration curves overlap somewhat at each end. In better grade instruments, the coils are wound of stranded wire to minimize the change of inductance with frequency and the condenser is placed in a shielded container to which one set of plates is grounded so that the capacitance of the condenser to walls of the room, body of the observer etc., is eliminated.

180. Relation between Wave Length and Frequency.—In any wave train there is a simple relation connecting the length of the wave, the velocity of propagation and the frequency of oscillation

of the particles of the medium through which the wave is progressing. For example, suppose Fig. 125 represents a train of waves traveling in the direction of the arrow with a velocity V. The distance between any two consecutive points in the same phase of vibration, such as A and B or C and D, is called the wave length λ. It is obvious that while the particle A executes one complete oscillation, the wave travels the distance AB. Accordingly, if f is the number of complete oscillations in one second and λ is the wave length, the velocity of propagation V is given by

$$V = f\lambda \tag{2}$$

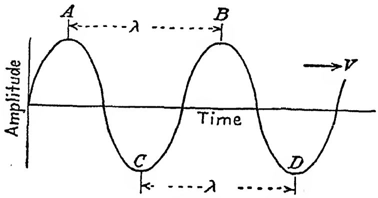

FIG. 125.—Simple wave train.

Thus, if the velocity of propagation, which for radio waves is that of light, 3×10^{10} cms. per second, and the frequency are known, the wave length is given by

$$\lambda = \frac{V}{f} \tag{3}$$

Substituting the values of f from (1) in (3) we have

$$\lambda = 2\pi V\sqrt{LC} \tag{4}$$

If the fundamental units for inductance and capacitance, i.e., henries and farads respectively, are used, λ is given in centimeters, i.e.,

$$\lambda_{m}^{\circ} = 6\pi \,.\, 10^{10}\sqrt{LC} \tag{5}$$

It is, however, customary to express λ in meters, L in microhenries (μh) and C in micro-microfarads ($\mu\mu f$). Converting to these units becomes

$$\lambda_{m} = 1.884\sqrt{L(\mu h) \,.\, C(\mu\mu f)} \tag{6}$$

Since the degree of interference between two radio stations depends upon the difference between their respective frequencies rather than their wave-length separation, government authorities assign the frequency upon which a station is to operate instead of the wave length. For this purpose, the kilocycle, equal to

1,000 cycles per second, is the unit used, and in the broadcasting-frequency range the separation between stations is, at present, 10 kilocycles. To convert wave length in meters to frequency in kilocycles use the relation

$$f_{KC} = \frac{3.10^5}{\lambda_m} \tag{7}$$

181. Experiment 37. *Calibration and Use of a Frequency Meter.* This experiment consists of three parts:

(*a*) Measurement of an unknown inductance.

(*b*) Construction of an improvised frequency meter and calculation of its calibration curve from known values of L and C.

(*c*) Checking computed values of improvised frequency meter against a standard-frequency meter.

Connect the apparatus as shown in Fig. 126 in which A is a source of continuous oscillations of variable frequency, e.g., a vacuum tube driven oscillator. B is the improvised frequency meter to be calibrated and D is a standard-frequency meter.

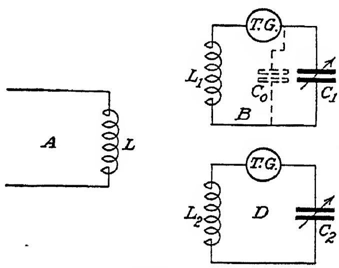

Fig. 126.—Calibration of frequency meter.

Leads in all circuits should be as short and direct as possible. Remove both B and D to a safe distance and put the circuit driver A in operation. Cautiously bring B into inductive relation with A. Swing C_1 slowly through its range of settings and note its reading when the thermo-galvanometer indicates a maximum. Use as loose coupling as possible, and in no case more than is necessary to give one quarter full scale reading at resonance. Extreme care must be used in getting this first setting, as an excessive current through the galvanometer will ruin it. Make several settings on C_1 for maximum current and average. This is the setting for circuit B to be in resonance with A. Remove B and measure the frequency of the oscillator with the meter D using the same precautions as noted above regarding coupling and scale deflections. In a similar manner check the improvised frequency meter at eight or ten points uniformly distributed across the scale.

After completing these measurements, compute the inductance of the coil of the improvised frequency meter. This may

readily be done by use of the equation (6) when the values of the capacitance are known. A calibration curve for the variable condenser of the improvised meter will be furnished. A correction must be made, however, for the distributed capacitance of the coil. When the frequency meter is in operation, one end of its coil is at a different potential from the other end and there is consequently some electro-static flux between the two halves of the coil equivalent to an additional capacitance C_0 in parallel with C_1 as shown by the dotted lines in Fig. 126. Instead of (6) we should write

$$\lambda = 1.884\sqrt{L(C + C_0)}. \tag{8}$$

C_0 may be calculated by substituting corresponding measured values of λ and C in (8) and solving for C_0. A sufficiently accurate graphical method is more convenient. If λ^2 is plotted against C as shown in Fig. 127, a straight line is obtained; and this, when extrapolated backward, cuts the axis of abscissas at C_0. Using this value of C_0, substitute corresponding values of λ and C in (8) and solve for L. Using now the average value of L thus obtained, recompute the values of λ as a function of C and plot a calibration curve for the improvised frequency meter.

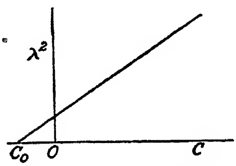

FIG. 127.—Graphical method for distributed capacitance of a coil.

Report.—Tabulate data and computations in compact form. Plot two calibration curves, (*a*) computed, (*b*) measured, on the same sheet, using wave length as ordinates and condenser settings as abscissas. For one of the curves compute the corresponding frequencies from equation (7) and plot on the same sheet kilocycles against condenser settings. How do you account for the discrepancy between observed and calculated values for wave length? Why is it important to remove or detune one resonating circuit while making settings with the other?

182. Resistance at High Frequencies.—At ordinary frequencies, such as are used for power and light purposes, the resistance of a conductor is practically independent of frequency and has the same value as for direct currents. At radio frequencies, however, this is not true. One of the chief causes of resistance variation with frequency is the so-called "Skin Effect." The outer portion of a wire carrying a current is surrounded by a smaller magnetic flux than the inner portion and consequently

offers a smaller impedance to a rapidly alternating current; and since the current distributes itself inversely as the impedance, the current density is greater nearer the surface of a conductor than at its interior. In fact, at sufficiently high frequencies, the interior carries no current at all. The result is a reduction in the effective cross-section of the conductor causing an increased resistance. For tables, see circular Bureau of Standards, No. 74, pp. 309–311.

Again the individual turns of a coil act as condenser plates for which the insulation on the wires and the supporting members furnish the dielectric. During each cycle of the current, the electric flux through these materials is reversed and a hysteresis phenomenom, similar to that for the magnetic flux in iron, results in an energy consumption equivalent in its effect to an increase in the resistance of the coil. This effective resistance is proportional to the frequency.

A third source of equivalent resistance is energy consumption due to eddy currents. At radio frequencies, relatively large E.M.F.s are induced in surrounding bodies and, even though their conductivities may be small, considerable energy may be consumed in this way. This also appears as an added resistance to the coil, proportional to the square of the frequency.

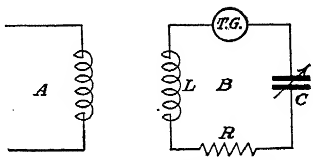

FIG. 128.—Resistance variation method.

183. Measurement of Resistance at High Frequency.—The simplest method of measuring resistance at radio frequencies is the "Resistance Variation Method," and is illustrated by Fig. 128 where A is a circuit driver of variable frequency and L the coil whose resistance is to be measured. C is a variable "low loss" tuning condenser whose calibration need not be known, and R a variable resistance of low-distributed inductance and capacitance. $T.G.$ is a thermo-galvanometer for indicating the resonance condition. Suppose the circuit B to be tuned to resonance with A, with the variable resistance R set at zero. Since the reactances neutralize each other at resonance the current I_0 in B is given by

$$I_0 = \frac{E}{R_0} \tag{9}$$

where E is the E.M.F. induced in L and R_0 is its resistance.

Without making any other change in the circuits, introduce a resistance R_1 in R. The current in B falls to a new value I_1, given by

$$I_1 = \frac{E}{R_0 + R_1} \tag{10}$$

Since the reading of the thermo-galvanometer is proportional to the square of the current through it, we have, called d_0 and d_1, the readings corresponding to I_0 and I_1 respectively.

$$d_0 = KI_0{}^2 = K\frac{E^2}{R_0{}^2}$$

$$d_1 = KI_1{}^2 = K\frac{E^2}{(R_0 + R_1)^2}$$

whence

$$\frac{d_0}{d_1} = \frac{(R_0 + R_1)^2}{R_0{}^2}$$

Taking the square root of each side and solving for R_0 we have

$$R_0 = \frac{R_1}{\text{[illegible]}} \tag{11}$$

184. Experiment 38.—*Measurement of Resistance at High Frequencies.*—Connect the apparatus as shown in Fig. 128 and set the circuit driver in oscillation at a suitable frequency as measured by a frequency meter. Tune the measuring circuit to resonance and adjust the coupling so that the thermo-galvanometer indicates about two-thirds full-scale deflection with R set at zero. The coupling between A and B should be loose enough so that when R is changed from zero to its maximum value no appreciable change occurs in the current in the oscillatory circuit of the circuit driver. Insert several different resistances in R and read the corresponding indications of the galvanometer. Compute R_0 by eq. (11) using several different combinations of the readings and take the average. Repeat using eight or ten different values of frequencies covering as wide a range as can be secured from the circuit driver. Measure the D.C. resistance of the coil by a Wheatstone bridge, also that of the thermo-galvanometer, and subtract the latter from the values obtained above.

Report.—Plot a curve showing the relation between the effective resistance of the coil and frequency, starting with the D.C. or zero frequency value. In what way would an error be introduced if the coupling between A and B were too tight?

185. Resonance Phenomena.—When a mechanical system which possesses the necessary inertia, stiffness, and small degree of friction, to make it capable of executing vibrations, is acted upon by an alternating force having a frequency equal to that of the system itself, oscillations of large amplitude result. If, however, the frequency of the impressed force differs somewhat from that of the driven system, the amplitude is much less than when the frequencies are the same. This selective response of the vibrating system is called "Resonance."

Electrical systems possessing inductance, capacitance, and sufficiently small resistance are also capable of oscillations and exhibit the phenomenon of selective response when excited by E.M.F.'s having frequencies equal to or nearly equal to their own fundamental frequencies. Resonance phenomena are of extreme importance in radio work and form the basis of all the experiments described in this chapter. Electrical resonance circuits are of two distinct types depending upon the manner in which the E.M.F. is applied to them, and are known respectively as "Series" and "Parallel" resonance circuits. In the former, the E.M.F. is introduced in series with the inductance and capacitance, while in the latter, the inductance and capacitance are connected in parallel across the source of E.M.F. In both cases large currents flow through the inductance and capacitance at resonance. In the series circuit, the impedance at resonance is small, equal in fact to the ohmic resistance of the circuit, while in the parallel circuit the impedance at resonance is large, and may be many times its ohmic resistance. Thus in transmission lines and filters series resonance circuits are used in the line when it is desired to pass currents of a certain frequency to the exclusion of other frequencies, and parallel resonance circuits when it is desired to suppress a particular frequency. The mathematical theory of these two types of resonance circuits is given above in articles 108 to 110 and the variation of current with frequency is shown in Figs. 79 and 82 respectively. A somewhat more detailed description of the reactances will, however, be given here.

Series Resonance.—A series resonance circuit consisting of a resistance, inductance, and capacitance is shown in Fig. 129*a* driven by an alternator E. Taking current as the basis of phase, the potential difference across the resistance and reactances have phase positions as indicated by the arrows. The potential difference across the resistance RI is in phase with the current.

Since the current through an inductance lags 90° behind the E.M.F., the potential difference, $L\omega I$, across the inductance leads the current by 90°. Similarly, the potential difference, $I/C\omega$, across the condenser lags 90° behind the current since the current through a condenser leads its voltage by that angle. The same current, of course, flows through all elements of the circuit.

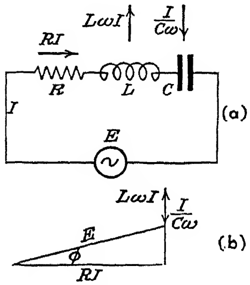

FIG. 129.—Vector relations for series resonance.

The voltages combine vectorially to give the total voltage across the circuit as shown in Fig. 129*b*. Here $L\omega I$ is measured vertically upward from RI, while $I/C\omega$ is measured downward from the top of $L\omega I$. Hence

$$E^2 = R^2I^2 + \left(L\omega - \frac{1}{C\omega}\right)^2 I^2 \text{ and } \phi = \tan^{-1} \frac{L\omega - \frac{1}{C\omega}}{R} \tag{12}$$

$$I = \frac{E}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}} \tag{13}$$

The current lags behind or leads the E.M.F. according as $L\omega$ is greater or less than $1/C\omega$, and is in phase with the E.M.F. when $L\omega = 1/C\omega$. The way in which the reactances $X_L = L\omega$ and $X_C = 1/C\omega$ change with frequency is illustrated by (*a*) in Fig. 130. X_L varies linearly with ω, while the curve for X_C is an equilateral hyperbola. The resultant reactance X is the sum of these two, shown by the full line. At the frequency ω_0 the two reactances neutralize each other, and the current is determined by the resistance alone. The variation of current with frequency is shown in (*b*) of the figure. The resonance frequency is determined by the relation

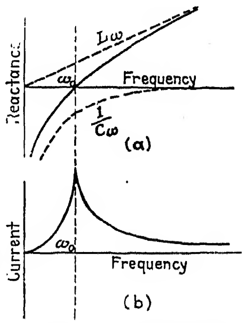

FIG. 130.—Reactance and current in series resonance.

$$L\omega_0 \quad \frac{1}{C\omega_0} = 0 \quad \text{or } \omega_0 = \quad LC \tag{14}$$

It is important to notice that the voltage drop across L and C individually may be larger than the impressed voltage E.

Parallel Resonance.—Such a circuit is shown in Fig. 131*a*. For the sake of simplicity we will assume that the resistance R is small enough so that the voltage drop across it may be neglected in comparison with that across the reactances X_L and X_C. Since the voltage across L and C is the same, it is convenient to take it as the basis of phase, and the angular positions of the currents are then as represented in Fig. 131*b*. $I_C = EC\omega$ leads the voltage by 90°, while $L_L = E/L\omega$ lags behind it by 90°. The resultant current I is the difference of these two and is given by

$$I = I_L - I_C = E\left(\frac{1}{L\omega} - C\omega\right) \tag{15}$$

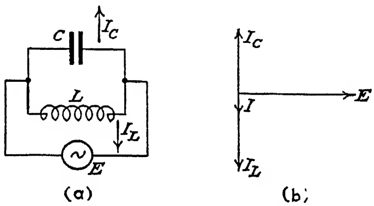

FIG. 131.—Vector relations in parallel resonance.

The resultant current for this simple case, $R = 0$, is thus always 90° out of phase with the voltage and may either lag or lead. This expression for I may be obtained by putting $R = 0$ in eq. 57, page 145, where the exact treatment is given. It is evident from eq. 15 that when $1/L\omega = C\omega$, $I = 0$. This is the parallel resonance condition and the resonance frequency ω_0 is given by

$$\frac{1}{L\omega_0} - C\omega_0 \quad 0 \quad \text{or} \quad \omega_0 = \frac{1}{\sqrt{LC}} \tag{16}$$

which is the same as for series resonance above. The accurate treatment, however, in which R is not neglected, shows that the resonance frequency for the parallel and series connections using the same values of L, C and R in each case is not quite the same. For most practical purposes, however, they may be taken as identical.

Plotting now the currents in the two branches as a function of impressed frequency, we have the curves shown in (*a*) Fig. 132. The current through the condenser increases linearly with the frequency while that through the inductance is inversely pro-

portional to the frequency. Computing the impedance Z from eq. (15) which in this case is made up of reactance only, we have

$$Z = \frac{E}{I} = \frac{1}{\frac{1}{L\omega} - C\omega} = \frac{L\omega}{1 - LC\omega^2} \tag{17}$$

The variation of impedance with frequency is shown in (*b*) Fig. 132. For this case, $R = 0$, Z is infinite at the resonance frequency. In the above reference, Z is shown to be L/CR, when R is not neglected.

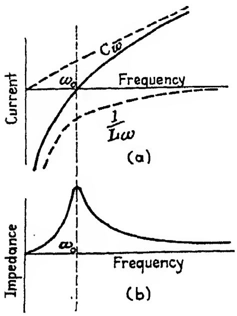

FIG. 132.—Impedance and current relations in parallel resonance.

186. Experiment 39. *Series and Parallel Resonance.*—The apparatus consists of (*a*) a circuit driver, i.e., a vacuum tube driven oscillator covering a reasonably wide range of frequencies and giving a fairly constant voltage throughout this range; (*b*) a resonating circuit of variable frequency covering about the same frequency range as the driver; (*c*) a frequency meter, and (*d*) a thermo-galvanometer or low-range ammeter. First calibrate the driver in the manner of Exp. 37 throughout its range and plot a curve showing its frequency for each setting of its variable condenser. Next replace the frequency meter by the resonance circuit and calibrate it in terms of the circuit driver using the same precautions regarding loose coupling, etc., as in the case of the frequency meter. To insure similarity of frequency range in the two circuits, set the condensers of both driver and resonator at their mid points and choose such a number of turns on the resonator inductance as to give approximate resonance.

Series Resonance.—With the apparatus connected as in Fig. 133, in which *C.D.* is the circuit driver and *R* the resonator, set *C* at its mid point and establish resonance by varying the frequency of the driver. Choose such a coupling that the thermo-galvanometer, *T.G.* indicates about eight-tenths full-scale deflection. Since the E.M.F. producing the current in the resonator is that induced in *L* by mutual inductance, i.e., in series with the circuit, this is a case of series resonance. Vary the frequency of the driver above and below this resonance frequency reading the

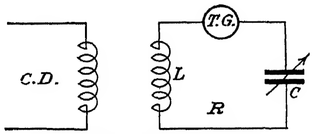

FIG. 133.—Connections for series resonance.

thermo-galvanometer for each setting, and take sufficient observations to plot a resonance curve. Introduce a few ohms resistance in the resonator circuit and repeat the observations. In this manner secure data to plot the family of resonance curves shown in Fig. 77, page 143.

Parallel Resonance.—Remove the added resistance from the resonator circuit and again bring the two circuits into resonance near the mid frequency as at the beginning of the series resonance experiment. Orient the coil L until its mutual inductance with the driver coil is zero and leave it in this position throughout this part of the experiment. Connect with the driver for parallel resonance as shown in Fig. 134, using the potential across one turn of the driver to energize the resonator. Insert the thermo-galvanometer 1, 2, and 3 to measure respectively the total current to the resonator, the current in the inductance, and the current in the condenser. Vary the frequency of the resonator

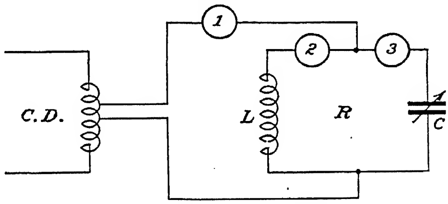

Fig. 134.—Connections for parallel resonance.

through the same range as in the case of series resonance and record the readings of the instruments. Again bring the circuits to resonance at the mid frequency, and take a similar series of observations varying the condenser of the resonator throughout its entire range, leaving the frequency of the driver constant.

Report.—Note that the deflection of the thermo-galvanometer is proportional to the square of the current; hence the square root of all readings must be taken to secure numbers proportional to the current. Plot calibration curves, i.e., frequency against condenser settings for both circuit driver and resonator. Plot the family of curves for series resonance, i.e., current against frequency; also the curves for parallel resonance, both for the case of fixed resonator constants with variable impressed frequency and constant impressed frequency with variable resonator frequency. Plot all three currents for each case of parallel resonance. Explain by vector diagrams the shape of all curves.

Why must mutual inductance between L and the circuit driver be zero for the parallel resonance case?

187. Resonance in Coupled Circuits.—When two circuits each having resistance less than the critical value and capable, therefore, of oscillating independently, are coupled together, the resulting system is doubly periodic; that is, it possesses two natural frequencies. These frequencies are different from either of the uncoupled frequencies, one of them being greater than the larger of the uncoupled frequencies, and the other less than the smaller of the uncoupled frequencies.

The reason for this change in frequency with coupling can be understood from the following consideration. In Fig. 135 A and B are two oscillatory circuits with reactances as indicated, coupled by mutual inductance M. Suppose them to have been

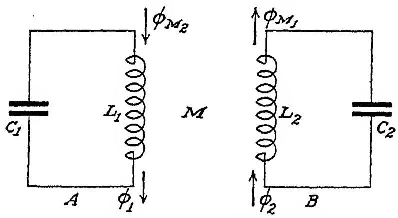

FIG. 135.—Two coupled circuits with mutual fluxes aiding.

set in oscillation by any appropriate means and that the currents in L_1 and L_2 are in such a direction as to produce fluxes ϕ_1 and ϕ_2 respectively as indicated by the arrows. Let ϕ_{M2} be the flux through L_1 due to the current in L_2 and ϕ_{M1} be the flux through L_2 due to the current in L_1. That these mutual fluxes are in the directions indicated may readily be seen by extending the magnetic flux lines of one coil around through the other in a closed path. Suppose the currents in A and B to have the same frequency and to maintain their given phase relation. It is obvious then that the effective inductances of L_1 and L_2 are each increased by the flux from the other. Thus by eq. (1) the frequency of each circuit is reduced.

Again, suppose that the current through one of the coils, B for example, is reversed. Then the mutual fluxes ϕ_{M1} and ϕ_{M2} have the directions indicated by the arrows in Fig. 136. It is seen that now the mutual flux is opposite in each case to the self-flux and the inductance of each coil is decreased, giving it a higher

frequency. If these circuits are suitably started, that is, each given current of the proper magnitude and phase relation, 0° or 180°, they will oscillate with one or the other coupled frequencies, depending upon the phase at starting. Such oscillations are spoken of as "Normal Modes." In general, when the system is set in oscillation by any random excitation, it oscillates according to both frequencies at once. Thus there are present in each circuit simultaneously two distinct frequencies, ω_1 and ω_2. Beats between these two frequencies result in alternate sum and difference amplitudes and we have the energy shifting from one circuit to the other at the frequency of the beats, i.e., $\omega_1 - \omega_2$.

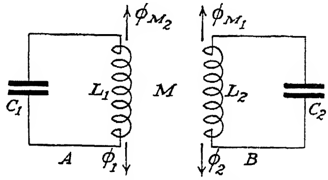

FIG. 136.—Two coupled circuits with mutual fluxes opposing.

It is easily shown[1] that if ω_a and ω_b are the uncoupled frequencies of the two circuits A and B and ω_1 and ω_2, the frequencies when coupled, that

$$\omega_1 = \sqrt{\frac{\omega_a^2 + \omega_b^2 + \sqrt{(\omega_a^2 - \omega_b^2)^2 + 4t^2\omega_a^2\omega_b^2}}{2(1 - t^2)}} \tag{18}$$

$$\omega_2 = \sqrt{\frac{\omega_a^2 + \omega_b^2 - \sqrt{(\omega_a^2 - \omega_b^2)^2 + 4t^2\omega_a^2\omega_b^2}}{2(1 - t^2)}} \tag{19}$$

where $t^2 = \frac{M^2}{L_1L_2}$ = coefficient of coupling.

$$\omega_a^2 = \frac{1}{\sqrt{L_1C_1}} \text{ and } \omega_b^2 = \frac{1}{\sqrt{L_2C_2}} \tag{20}$$

An interesting simplification results for the important special case in which the two circuits are tuned to the same frequency before coupling. Putting $\omega_a = \omega_b = \omega_0$ in the above expressions, there results

$$\left\{\begin{aligned} \omega_1 &= \frac{\omega_0}{\sqrt{1 - t}} \\ \omega_2 &= \frac{\omega_0}{\sqrt{1 + t}} \end{aligned}\right. \tag{21}$$

If such a coupled system, consisting of two circuits of identical uncoupled frequencies, is excited by a source of variable frequency, it is found to have two resonance frequencies, ω_1 and ω_2 as given above, and the separation between them increases with

[1] PIERCE; Electric Waves and Electric Oscillations.

tightness of coupling. Fig. 137 shows the resonance curves for such a system, the current in one of the circuits, B, being plotted against frequency. The degree of coupling here increases in the order 1 – 4.

Curve 1 shows only one peak but if sufficiently sensitive instruments were used, two peaks very close together would be found in this curve also. The coupling coefficient may easily be found from eq. (21). Squaring and dividing one by the other we have

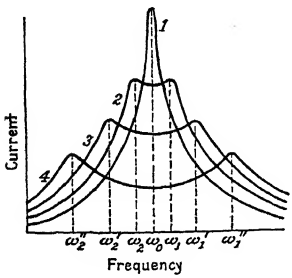

FIG. 137.—Resonance curves for two coupled circuits.

$$\frac{\omega_1^2}{\omega_2^2} = \frac{1+t}{1-t}$$

whence

$$t = \frac{\omega_1^2 - \omega_2^2}{\omega_1^2 + \omega_2^2} \tag{22}$$

188. Experiment 40. *Resonance in Coupled Circuits.*—Connect the apparatus as in Fig. 138 in which $C.D.$ is a circuit driver capable of giving a reasonably constant voltage over a considerable range of frequencies. A and B are two resonance circuits, preferably identical, having a variable mutual inductance between them. First, with both A and B removed, calibrate the circuit driver over its entire range of frequency by means of a frequency meter and plot frequencies against condenser settings. Set $C.D.$ at its middle frequency and adjust A and B independently to resonance with this frequency using a thermo-galvanometer to indicate resonance. Next couple A and B loosely and put the thermo-galvanometer in either circuit, and bring the system into inductive relation with $C.D.$ Adjust the coupling between the system and $C.D.$ so that the thermo-galvanometer reads about eight-tenths full-scale deflection. Now vary the frequency of the circuit driver taking readings on the thermo-galvanometer to obtain data for curve (1) of Fig. 137. Tighten the coupling and repeat successively for curves (2), (3) and (4).

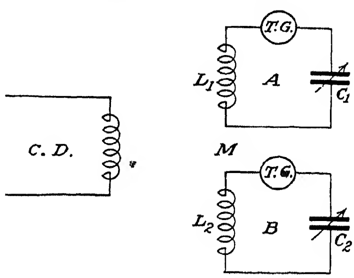

FIG. 138.—Circuit for resonance in coupled systems.

Report.—Plot a frequency calibration curve for the oscillator and the resonance curves as indicated. Compute the coupling coefficient for the different settings from formula (22). Explain the analogy between this electrical system and any mechanical system with which you are acquainted.

189. The Simple Antenna—Inverted L Type.—In the circuits previously considered, the inductance and capacitance were considered as localized at definite points in the circuit, and the terms "lumped inductance" and "lumped capacitance" are often used to distinguish circuit elements of this type from those we are to consider here. Circuits with lumped elements are characterized by the fact that ammeters inserted at various points in the circuit indicate that the same current flows at all points.

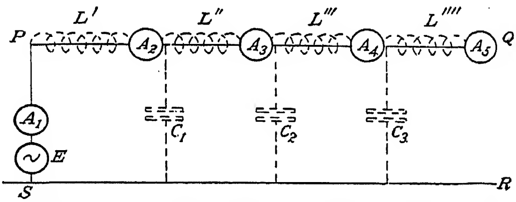

FIG. 139.—An antenna is a circuit with distributed inductance and capacitance.

Consider now an antenna of the inverted L type as shown in Fig. 139. Each element of length possesses a definite inductance because it is a part of a single-turn coil $PQRS$, supposing R and S to be joined by a conductor, and it also has a definite capacitance, some portion of the ground being the other element of the condenser. Let these be designated by L' . . . L'''' and C_1 . . . C_3 respectively. Let E be a source of E.M.F. of any desired frequency. It is obvious that at sufficiently high frequency the ammeters A_1 . . . A_5 will not indicate equal currents, since A_4 for example, measures the current flowing in C_3 only; A_3 that in C_2 and C_3, while A_2 indicates the current flowing in C_1, C_2 and C_3. A_5 reads zero since it is at the end of the antenna. Moreover, if voltmeters were connected between the antenna and ground at various points, they would indicate that the potential difference between the antenna and ground varies from point to point.

It is obvious that the current in the antenna varies from point to point along it because of the charging currents through the capacitances to ground, and that the potential also varies from

point to point because of the resistance and reactance of the flat-top portion through which the variable current is flowing. When the reactances are thus spread out forming a circuit in which currents of different magnitudes flow through the various parts, it is customary to speak of them as "distributed" inductances and capacitances as distinguished from the corresponding localized quantities previously studied.

In discussing the characteristics of antennas, it is advantageous to express the distributed inductance, capacitance, and resistance in terms of their effective values, that is to determine an equivalent circuit having values of lumped inductance, capacitance, and resistance such that when joined in series the same current would flow as actually flows through the antenna circuit at the point of power supply. To compute the equivalent values for such a phantom circuit,

let L_1, C_1 and R_1 = inductance, capacitance, and resistance per unit length of antenna

l = length of horizontal part of antenna

L_0, C_0 and R_0 = $L_1 l$, $C_1 l$ and $R_1 l$ respectively.

Thus L_0 is the inductance of the single-turn loop $PQRS$, supposing R and S to be joined, when a current is flowing through the loop having a frequency so low that the same current flows in all parts of the circuit. C_0 is the capacitance of the condenser formed by the antenna and ground when the voltage is distributed uniformly over the antenna. It is possible to express the effective values L_e and C_e of the distributed inductance and capacitance, respectively, in terms of L_0 and C_0, the low-frequency or lumped values, by simple relations which are sufficiently accurate for most purposes.

By[1] an application of the transmission line formula to the flat top portion of an antenna, it may be shown that its reactance, when supplied with an alternating E.M.F. at one end is

$$X = -\sqrt{\frac{L_1}{C_1}} \cot \omega l \sqrt{L_1 C_1} \tag{23}$$

or

$$X = -\sqrt{\frac{L_0}{C_0}} \cot \omega \sqrt{L_0 C_0} \tag{24}$$

where ω is the radian frequency of the applied E.M.F. For low frequencies, that is for frequencies small compared with the

[1] Miller, *Proc. Inst. Radio Engrs.*, Vol. 7, pp. 299, 1919.

natural frequency of the antenna, the above expression may be put in a simple approximate form. Since

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \cdots$$

$$X = -\sqrt{\frac{L_0}{C_0}}\left(\frac{1}{\omega\sqrt{L_0C_0}} - \frac{\omega\sqrt{L_0C_0}}{3} - \frac{(\omega\sqrt{L_0C_0})^3}{45} - \cdots\right) \quad (25)$$

or

$$X = -\frac{1}{C_0\omega} + \frac{L_0\omega}{3} \text{ approximately} \quad (26)$$

for small values of ω. Equation 26 is the reactance for a simple series circuit having a capacitance C_0 and an inductance $L_0/3$. It thus appears that the reactance of an antenna with distributed inductance and capacitance, when operated at frequencies well below its natural frequency, is approximately the same as that of a series circuit with lumped capacitance C_0 equal to that of the antenna for uniform voltage distribution, and an inudctance equal to one-third that of the antenna supposing it to be a closed loop and operated at a frequency so low as to give uniform current distribution through out.

190. Calculation of the Wave Length of a Loaded Antenna.—Antennas are usually operated at frequencies varying from one-third to one-fifth of their natural frequency, and accordingly an inductance is inserted in the down lead for tuning purposes. This coil, which is thus in series with the inductance of the antenna, may also serve to introduce the power to the antenna through mutual inductance or other appropriate means of coupling it to the transmitter circuit. The frequency of the loaded antenna is then given by

$$f = \frac{1}{2\pi\sqrt{\left(L + \frac{L_0}{3}\right)C_0}} \quad (27)$$

where L is the inductance of the loading coil. In this formula, inductance and capacitance are expressed in henries and farads respectively. The wave length is given by

$$\lambda = 1{,}884\sqrt{\left(L + \frac{L_0}{3}\right)C_0} \quad (28)$$

In eq. (28) the constant 1,884 has been so chosen that λ is in meters when the inductance and capacitance are expressed in micro-henries and micro-farads respectively.

The low-frequency inductance of a single straight round wire is given by the following formula taken from Bureau of Standards Circular, No. 74, page 243.

$$L_0 = 0.002l\left[2.303 \log_{10}\frac{4l}{d} - 1 + \frac{\mu}{4}\right] \text{ micro-henries} \qquad (29)$$

where l = length of wire in centimeters.
d = diameter of cross-section in cms.
μ = permeability of the material of the wire.

The low-frequency capacitance of a single wire of length l cms. and diameter d cms. suspended h cms. above the ground is given by one or the other of the following formulae depending upon the relative values of h and l. For

$$\frac{4h}{l} \lesseqgtr 1,\ C_0 = \frac{0.2416l}{\log_{10}\frac{4h}{d} - K_1} \text{ micro-microfarads} \qquad (30)$$

For

$$\frac{4h}{l} \gtreqless 1,\ C_0 = \frac{0.2416l}{\log_{10}\frac{2l}{d} - K_2} \text{ micro-microfarads} \qquad (31)$$

in which

$$K_1 = \log_{10}\left[\frac{1 + \sqrt{1 + \left(\frac{4h}{d}\right)^2}}{2}\right] \qquad (32)$$

and

$$K_2 = \log_{10}\left[\frac{l}{4h} + \sqrt{1 + \left(\frac{l}{4h}\right)^2}\right] \qquad (33)$$

These formulae assume that the distribution of charge from point to point along the wire is uniform. Tables for K_1 and K_2 for a series of values of $\frac{4h}{l}$ are given in the appendix, table IV.

191. Experiment 41. *Calculation of the Wave Length of a Loaded L Type Antenna.*—Obtain from the instructor data for the length and height of the antenna. Measure the diameter of the wire by a micrometer. Use as a loading coil a single-layer solenoid provided with clips for picking off any desired number of turns. Compute by means of formula (29) the inductance L_0 of the antenna and its capacitance from (30) or (31) as the case may be. Measure the dimensions of the loading coil and compute from formula (1) page 306 its inductance for a series of five different numbers of turns. By means of formula (28) compute the wave length of the antenna when loaded with these

inductances. Connect these inductances successively in series with the antenna including a thermo-galvanometer as an indicating instrument. Energize the antenna weakly by a circuit driver, i.e., a vacuum-tube oscillator of variable frequency, and measure the wave length for each of the series loading coil turns for which the inductance has been computed.

Report.—Tabulate your computed values for the inductance and capacitance of the antenna and the inductance of the loading coil, making clear the various steps. Plot curves for the computed and measured values of the wave length as a function of the loading coil turns. Try to account for the discrepancy between computed and measured values by a consideration of such approximations as neglect of inductance and capacitance of down lead, proximity of buildings and trees, etc.

192. Measurement of Inductance and Capacitance of an Antenna.—It was pointed out in Art. 190 that when an antenna is used with a loading coil in series with the down lead so that the frequency at which it is operated is considerably less than its natural frequency, that the wave length may be computed with fair accuracy from the quantities L_0 and C_0, which represent respectively the low-frequency inductance and capacitance of the antenna. The effective values of inductance and capacitance at radio frequencies L_e and C_e are connected with the low-frequency values by the relations

$$L_e = \frac{L_0}{3} \text{ and } C_e = C_0.$$

If now the wave length of the antenna is measured for two different loading inductances L_1 and L_2, then by eq. 28, we have

$$\lambda_1 = 1{,}884\sqrt{(L_1 + L_e)C_e} \tag{34}$$

$$\lambda_2 = 1{,}884\sqrt{(L_2 + L_e)C_e} \tag{35}$$

Squaring both sides of these equations and dividing, we have

$$\frac{\lambda_1^2}{\lambda_2^2} = \frac{L_1 + L_e}{L_2 + L_e}$$

whence

$$L_e = \frac{L_1\lambda_2^2 - L_2\lambda_1^2}{\lambda_1^2 - \lambda_2^2} \tag{36}$$

Substituting this value of L_e in either (34) or (35), preferably the one with the larger loading inductance, L_2 for example, we have, after simplifying

$$C_e = \frac{\lambda_2^2 - \lambda_1^2}{(1{,}884)^2(L_2 - L_1)} \tag{37}$$

Thus by measuring the wave length of the antenna for two different loading inductances, it is possible to determine both the effective inductance and capacitance in terms of these loading inductances and the correspond ng wave lengths.

193. Effective Resistance of an Antenna.—The effective resistance of an antenna is a somewhat complicated quantity and includes much more than the ordinary ohmic resistance of the wires of which it is constructed. When electrical energy is converted into any other form, it is convenient, for many purposes, to reduce it to the equivalent heat-development basis, no matter what the form may be. Thus a storage battery, while being charged, may be said to be equivalent to a resistance R where R is so chosen that, when carrying a current equal to the charging current of the battery, the same number of joules of heat are liberated in it per second as are consumed per second in the battery. In a similar manner the effective resistance of an antenna is defined by the equation

$$I^2R_e = W \tag{38}$$

where I is the antenna current at the point where the driving E.M.F. is introduced and W is the power in watts delivered to the antenna.

This hypothetical resistance is made up of as many different parts as there are forms of energy dissipation in the antenna system. Of these various forms, only that which is converted into energy of radiated electromagnetic waves is useful. The others are wasteful and must be kept low for good efficiency. The "Radiation Resistance" R_r as it is called, is defined as a fictitious resistance such that if inserted at the point where the current is a maximum, it would dissipate energy in the form of heat at the same rate as the antenna actually radiates it in the form of electromagnetic waves. The radiation resistance is not a constant quantity but is proportional to the square of the effective height and inversely proportional to the square of the wave length on which the antenna is operated. It is represented as a function of the wave length in Fig. 140.

A second form of energy dissipation is the obvious one due to ohmic resistance R_0 in overhead wires, down lead, counterpoise, if one is used, etc. Because of the variations in the current distribution in conductors due to skin effect, and eddy currents induced in the ground and nearby conductors, the ohmic resistance varies somewhat with the frequency, being greater at the

shorter wave lengths. These variations are appreciable only at very high frequencies and the ohmic resistance of the antenna is represented as the straight line R_0 in Fig. 140.

The third cause of energy loss in an antenna is due to dielectric hysteresis. The insulators on which it is suspended, trees, buildings, and other non-conducting or poorly conducting bodies form part of the insulating material between the antenna and ground, and their polarizations are reversed with each alternation of the antenna current. This represents an energy loss similar to that due to hysteresis in iron. This loss, which varies linearly with the wave length, is represented by the straight line marked R_d in the figure. The total effective resistance of an antenna is the sum of these separate resistances and is indicated by the dotted line marked R_e, which is a typical resistance curve for an average antenna. Frequently the resistance curve shows marked humps at certain frequencies. These are usually due to energy absorption by resonating electric systems such as supporting masts or towers, guy wires not sufficiently broken by strain insulators, metal beams in nearby buildings, plumbing pipes, etc. When antennas are operated at excessively high voltages, there may be additional energy losses due to brush discharge from the wires. This loss was more common with the old spark transmitters than with present-day continuous wave equipment.

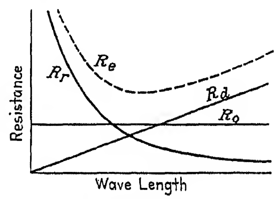

FIG. 140.—Resistance of an antenna at different wave lengths.

194. Measurement of the Effective Resistance of an Antenna. The effective resistance of an antenna may readily be measured by the resistance variation method of Exp. 38. If an antenna with its loading inductance is operated at resonance frequency by a suitable circuit driver, the current in it is given by

$$I = \frac{E}{R_e} \tag{39}$$

If a resistance R_1 is inserted in series with it and no other change is made, the current becomes

$$I_1 = \frac{E}{R_1 + R_e} \tag{40}$$

If these currents are measured by a thermo-galvanometer, or any other instrument in which the deflection is proportional to the

square of the current, the ratio of the currents, obtained by dividing (39) by (40), is

$$\frac{I}{I_1} = \frac{R_1 + R_e}{R_e} = \sqrt{\frac{d_1}{d_2}} \tag{41}$$

where d_1 and d_2 are the readings of the instrument corresponding to I_1 and I_2. Solving (41), we have

$$R_e = \frac{R_1}{\sqrt{\frac{d_1}{d_2}} - 1}$$

195. Experiment 42. *Measurement of the Effective Inductance, Capacitance, and Resistance of an Antenna.*—Connect the apparatus as shown in Fig. 141 where A is the antenna to be studied; R a variable resistance with low distributed inductance and capacitance; L a tuning coil whose inductance for several different taps is known; $T.G.$, a thermo-galvanometer; $C.D.$, a circuit driver of variable frequency, and $W.M.$ a wave meter. Set $R = 0$, and L at a value near its maximum. Move the wave meter to a distance sufficient to prevent its absorbing energy from the system. Couple the circuit driver loosely to the antenna and adjust the frequency until the antenna resonates to it as indicated by the thermo-galvanometer. Reduce the coupling until the thermo-galvanometer deflects only a few divisions at resonance. Open the antenna circuit and measure the frequency of the oscillator by the wave meter. Repeat the process, using five other values of L. Compute the effective inductance and capacitance of the antenna by formulas 36 and 37 using different combinations for L_1 and L_2.

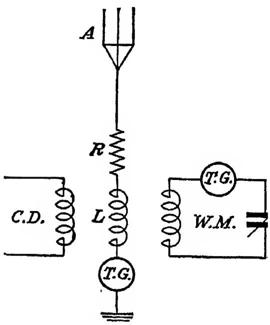

FIG. 141.—Connections for measuring antenna constants.

Measure the effective resistance of the antenna for each of the values of L used above. For this purpose, couple the circuit driver closely enough so that the thermo-galvanometer indicates about two-thirds full-scale deflection with $R = 0$. Increase R until the deflection has been reduced to about one-half its original value. Repeat for three other values of R. Compute R_e by formula (42). The circuit driver should have sufficient power to permit coupling with the antenna loose enough so that when R is

changed through the necessary range for these measurements, the ammeter in its oscillatory circuit does not show an appreciable variation.

Report.—Tabulate all results in compact form. Plot a curve showing the resistance of the antenna as a function of wave length. Why will an error be introduced if the current in the circuit driver varies when R is changed? Why should the wave meter be removed when taking observations on the antenna, and the antenna circuit opened when taking measurements of the wave length of the circuit driver?

CHAPTER XVI

ELECTRON TUBES

During the past two decades the electron tube has advanced to a position of great importance in the research laboratory and in industry, as well as in the art of communication. The present chapter will be devoted to its study.

196. Free Electrons.—It is a recognized fact that the passage of an electric current through a conductor is in reality a motion of electrons. When atoms or molecules are crowded closely together as in a solid or liquid, the electrons near the periphery of each atom are subjected to large forces due to the neighboring atoms so that their behavior is different than would be the case were the atom far from its neighbors as in a gas or vapor. In an electrical conductor such as a metal, one of the effects of these forces is to enable the valence electrons, or at least some of them, to become detached from the parent atom and move about within the metal in a more or less random manner. If an externally applied electric field is present, as when the two ends of a conductor are connected to a battery, these electrons will drift in response to the field thus constituting an electric current. The electrons partaking in this motion are commonly called "free" or "conduction" electrons.

Various theoretical and experimental investigations have shown that for the purpose of studying most phenomena the presence of the remaining parts of the atoms may be entirely neglected and we may consider the free electrons as constituting a sort of "electron gas" enclosed in a container whose walls correspond to the bounding surfaces of the metal. This electron gas absorbs part of the thermal energy of the metal and the manner in which this energy is distributed among the various electrons is much the same as the distribution of thermal energy among the molecules of a real gas.

Under normal conditions the free electrons are forced to remain within the boundaries of the metal by certain electrical forces existing at the surface. The exact nature of these forces

is not completely known but it is certain that the so-called image forces play an important role. If an electron travels slightly beyond the surface atoms, a positive charge is induced on the surface so that there is an attractive force tending to pull the electron back into the metal. Thus in order to completely escape from the metal the electron must perform a certain amount of work at the expense of its kinetic energy. This work, usually represented by W_a is a constant, at least for the particular metal.

Under normal conditions of temperature no electrons within the metal possess sufficient energy to enable them to overcome the attractive forces and consequently all of them are forced to remain within the metal. It is possible, however, by various means to impart sufficient energy to some electrons to enable them to escape. One method of doing this is to direct light of sufficiently high frequency upon the surface. This is the familiar photo-electric effect. Free electrons may also absorb energy from impinging ions or electrons in amounts sufficiently great to allow them to overcome the binding forces. The cathode rays of a highly evacuated gas discharge tube are secondary electrons arising from the bombardment of the cathode by positive ions.

By far the easiest method of obtaining emission of electrons from a metal is, however, to heat it and thus give some of the free electrons sufficient thermal energy to escape. It is this method in which we are primarily interested here.

197. The Richardson Equation.—Let us choose coordinates so that the surface of our metal lies in the yz plane. In order to determine the total number of electrons escaping each second from unit area of the surface we must calculate how many reach the surface each second with velocities V_x perpendicular to the surface sufficiently great to satisfy the relationship $mV_x^2/2 > W_a$.

Modern quantum theory shows that the number of electrons N per cc. possessing velocity components between V_x and $V_x + dV_x$, V_y and $V_y + dV_y$, V_z and $V_z + dV_z$ is given by

$$NdV_xdV_ydV_z = \frac{CdV_xdV_ydV_z}{(\epsilon^{-(W_i - W)/kT} + 1)} \tag{1}$$

where C and W_i are constants depending in part on the total number of free electrons, k is the Boltzmann gas constant (1.37 ergs per degree), ϵ is the natural log base, T is the absolute temperature, and $W = \frac{1}{2}mV^2 = \frac{1}{2}m(V_x^2 + V_y^2 + Vi_z^2)$ is the kinetic energy.

W_i has a particular significance as follows. Let us consider the particular case when $T = 0$. Then $N' = C$, if $W < W_i$, and $N' = 0$, if $W > W_i$. Thus when the metal is at absolute zero electrons are present at all energies up to W_i but no electrons have energy greater than W_i so that W_i may be considered the maximum kinetic energy that an electron may have at absolute zero. The distribution in this case is illustrated by the solid line in the graph of Fig. 142. For values of T greater than zero the distribution takes the form shown by the various dotted lines. That is, some electrons have energies greater than W_i, their number decreasing rapidly as we move toward higher energies. W_i is a constant for any given metal but varies from metal to metal, being proportional to the two-thirds power of the number of free electrons per cc.

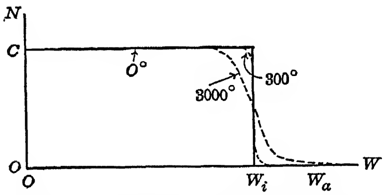

FIG. 142.—Velocity distribution of free electrons.

In all metals W_i is much less than W_a. Then in order to escape the electron must possess energy much greater than W_i. When this is true the exponential term in the denominator of eq. (1) becomes very large and we may neglect the 1 in comparison. Therefore, in the range in which we are interested

$$N'dV_xdV_ydV_z = C\epsilon^{(W_i - W)/kT}dV_xdV_ydV_z$$

Substituting for W

$$N'dV_xdV_ydV_z = C\epsilon^{W_i/kT}\epsilon^{-mV_x^2/2kT}\epsilon^{-mV_y^2/2kT}\epsilon^{-mV_z^2/2kT}dV_xdV_ydV_z$$

We are interested in all electrons of velocity V_x regardless of their other velocity components. To obtain the total number of such electrons we must integrate over the variables V_y and V_z. Then the total number N' of electrons having an x component of velocity in the range V_x to $V_x + dV_x$ is given by

$$N'dV_x = C^{W_i/kT}\epsilon^{-mV_x^2/2kT}dV_x\int_{-\infty}^{\infty}\epsilon^{-mV_y^2/2kT}dV_y\int_{-\infty}^{\infty}\epsilon^{-mV_z^2/2kT}dV_z$$

Now $\int_{-\infty}^{\infty}\epsilon^{-x^2} = \sqrt{\pi}$

Making use of this relationship

$$N'dV_x = C\left(\frac{2\pi kT}{m}\right)\epsilon^{W_i/kT}\epsilon^{-mV_x^2/2kT}dV_x \tag{2}$$

The rapidity with which electrons of velocity V_x strike each square centimeter of the surface is equal to the number of such electrons per cubic centimeter multiplied by V_x (see elementary gas theory). Then the number striking each square centimeter of our surface per second with this velocity will be given by

$$N'V_x dV_x = C\left(\frac{2\pi kT}{m}\right)V_x\epsilon^{W_i/kT} \cdot V_x\epsilon^{-mV_x^2/2kT}dV_x$$

All electrons striking the surface in such a way that $\frac{1}{2}mV_x^2 > W_a$ will be able to escape. Then to calculate the number of escaping electrons we must integrate the above expression over all values of V_x from $\sqrt{2W_a/m}$ to infinity. Each escaping electron carries a charge e so that the total charge per second, i.e., the current, will be

$$i = eC\left(\frac{2\pi kT}{m}\right)\epsilon^{W_i/kT}\int_{\sqrt{2W_a/m}}^{\infty} V_x\epsilon^{-mV_x^2/2kT}dV_x$$

$$i = eC\left(\frac{2\pi kT}{m}\right)\left(\frac{kT}{m}\right)\epsilon^{W_i/kT}\epsilon^{-W_a/kT} \tag{2a}$$

This equation is usually written

$$i = AT^2\epsilon^{-\phi e/kT} \tag{3}$$

where $A = 2\pi eC(k/m)^2$ and $W_a - W_i = \phi e$. Equation (3) is known as Richardson's thermionic equation in honor of O. W. Richardson, who first discovered it empirically and who has done much work in thermionics during the past three decades.

The quantity ϕ is called the thermionic work function and is usually measured in volts. The quantity ϕe is seen to be the energy by which the fastest electron falls short of being able to escape at a temperature of absolute zero.

Equation (3) may be written in another way. Dividing by T^2 and taking the log of both sides

$$\log\left(\frac{i}{T^2}\right) = \log A - \frac{\phi e}{kT} \tag{3a}$$

This gives us a means of determining the value of ϕ experimentally without knowing either A or the area of the metal surface. If we plot log (i/T^2) as ordinates against $1/T$ as abscissas, the result is a straight line of slope $-\phi e/k$. Hence knowing the slope of the curve and the values of k and e, we may calculate θ in volts.

198. The Two-element Electron Tube.—This device consists of a filament F (Fig. 143) which may be heated to any desired temperature by an electric current. Surrounding the filament is a metal cylinder called the plate. Electrons emitted by the filament are drawn to the plate by an electric field furnished by the battery B. On reaching the plate they enter it and pass through the external circuit, thus forming an electric current. In their passage from filament to plate the electrons constitute a so-called "space current."

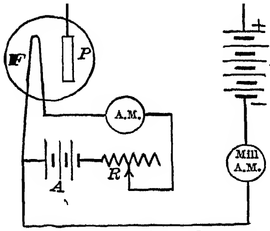

FIG. 143.—Two element electron tube.

199. Experiment 43. *Measurement of the Thermionic Work Function.*—Connect the tube as shown in Fig. 143*a*. Obtain from the instructor the proper values of the various potentials and currents. The sensitivity of the galvanometer G may be varied by changing the values of r_1 and r_2. If the sum of r_1 and r_2 is kept constant the current sensitivity of the whole is proportional to r_1.

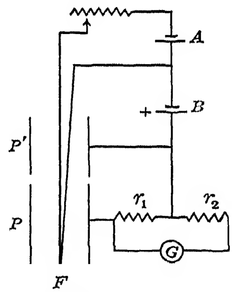

FIG. 143*a*.—Measurement of thermionic work function.

Measure i for about 10 different temperatures, obtaining the latter from the current-temperature curve furnished. Plot $\log_i (i/T^2)$ as ordinates against $1/T$ as abscissas. From the slope of the curve and the values of k and e ($e = 1.59 \times 10^{-19}$ coulombs) calculate the value of ϕ for the filament used.

Report.—(1) Why is the plate split into two parts? (2) Why is it necessary to use a fairly high plate potential? Note that this introduces an uncompensated (but quite negligible) factor in the development of Richardson's equation. ((3) Perform the integrations leading to eqs. (2) and (2*a*). (4) Show that the galvanometer sensitivity is directly proportional to r_1 provided $r_1 + r_2$ remains constant. (5) Determine from the values of ϕ listed in tables what the metal used in the filament is. (6) Plot curves similar to Fig. 142 from eq. (1) using temperatures of 0°, 300°, and 3000°.)

200. Voltage and Space Charge Saturation.—In Exp. 43 we have used a plate potential large enough to ensure that electrons

are being drawn to the plate as rapidly as they are emitted from the filament. This is not always the case. When the filament is hot it is at all times surrounded by a cloud of electrons known as the "space charge." These electrons produce a field tending to drive further electrons back to the filament as soon as they are emitted. If the plate were not present the region around the filament would soon be filled to a definite density depending upon the temperature of the filament and an equilibrium state would be reached in which the number returning to the filament would equal the number being emitted.

When a small positive charge is applied to the plate a few electrons are drawn to the latter and a small space current is thus set up. If, with the filament at a constant temperature T_1, this potential is gradually increased, electrons will be drawn to the plate more and more rapidly, resulting in an increase in space current. This increase cannot go on indefinitely, however, for soon a potential will be reached at which electrons will be drawn to the plate as rapidly as they are being emitted by the filament. The emission of the filament is thus a limiting factor upon the space current. If now the filament temperature is raised to a value T_2 and the process repeated the results will be as before but the maximum space current will be larger since the number of electrons emitted is greater than before. Under such circumstances the tube is said to be in a condition of "voltage saturation" and the maximum current is called the "saturation current."

Suppose that we now fix the plate potential at some convenient value and vary the filament temperature in small steps. At low temperatures no electrons are emitted. As the temperature is gradually increased, however, a small space current begins to flow which increases with increasing filament temperature. As before, however, the increase does not continue indefinitely. A point is soon reached at which the repulsive force exerted by the space charge becomes comparable to the attractive force of the plate and the increase in space current becomes less rapid. Ultimately the field due to the space charge becomes great enough to completely neutralize the field due to the plate potential and no more electrons can be emitted until some of those near the plate have entered it and thus reduced the space charge. Electrons from the filament will then replace those absorbed by the plate keeping the space charge at constant density. Beyond

this point the space current no longer increases. If now the plate potential is increased and the process repeated the maximum space current is larger than before and is reached at a somewhat higher filament temperature as a greater density of electrons in the space charge is necessary to offset the field due to the plate. This condition is known as "space charge saturation."

(We thus see that the space current in a two element tube is dependent upon the geometry of the tube, the temperature of the filament, and the potential applied to the plate. Such a tube, variously called the Fleming valve, Kenetron, and diode, finds its chief application as a rectifier of alternating current since current can flow through it in one direction only. Care must be taken that the plate should not be heated sufficiently by the energy absorbed from the impinging electrons to enable it too to become an emitter.)

201. Experiment 43. *Characteristics of the Two-element Electron Tube.*—Connect the apparatus as shown in Fig. 143. The purpose of this experiment is to obtain the two sets of characteristic curves illustrated in Figs. 144 and 144*a*. Ascertain from the

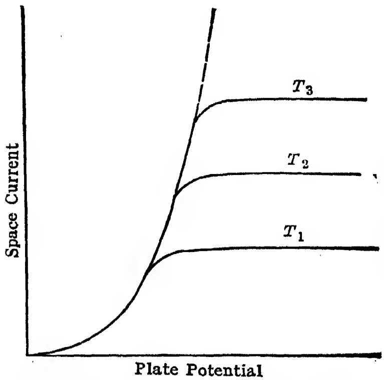

FIG. 144.—Voltage saturation curves.

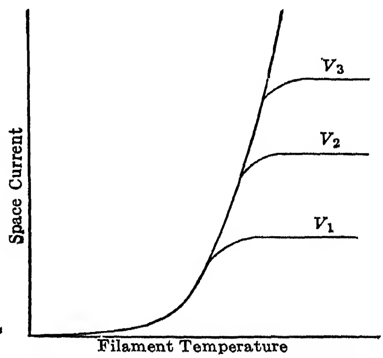

FIG. 144*a*.—Effect of space charge.

instructor the normal filament current for the tube, and using this and two smaller ones take plate potential-space current characteristics for each, varying plate potentials from zero to the maximum provided by your source. Determine also the filament current-space current characteristics for three different values of plate potential.

Report.—Plot the two sets of curves as indicated. Explain what is meant by voltage saturation and space charge. Why is it apparently impossible to obtain a completely saturated condition when varying the filament current with plate potential fixed?

202. The Three-element Electron Tube.—In the discussion of the two-element tube the dependence of space current upon filament temperature and plate potential was described, and it was pointed out that its principal application is in the rectification of high voltage alternating currents. By changing the temperature of the filament, thus regulating the supply of available electrons it also serves as a means of controlling currents. In this way, it acts as an electrical valve which may be opened or closed to any desired fraction of its current carrying capacity. Since, however, filament temperatures do not respond immediately to changes in heating current, this action is sluggish, and it can not be used in this way to produce current variations that are at all rapid.

It has been found that the space current may be controlled with remarkable ease by the introduction between the filament and plate of a third electrode in the form of a grid or mesh of fine wires through which the electrons must pass on their way from filament to plate. Such an arrangement is shown in Fig. 145. If a difference of potential is established between the filament and grid by means of the battery *C*, the grid tends to accelerate or retard the electrons of the space current according as it is positive or negative with respect to the filament. It thus counteracts or increases the effect of the space charge. The operation of the three-element tube may be best described by means of the curve of Fig. 146, which shows the relation between the plate current and the grid volts and is known as the "static characteristic." If the grid is disconnected from the circuit, the tube behaves as the ordinary two-element device in which the space current is limited either by electron emission of the filament or by space charge. Assuming there is available a sufficient supply of electrons so that the space charge is the controlling factor, a negative potential placed upon the grid adds to the retarding action of the space charge, and the plate current is reduced, and may even be made zero, if the grid is sufficiently negative. Again, if the grid is positive, it neutralizes to a certain extent the effect of the space charge, causing an increase of the space current. The space current can not continue increasing indefinitely, for even though

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the space charge were completely neutralized by positive charges on the grid, the current would be limited by the electron supply at the filament. This accounts for the horizontal part of the static characteristic. If a higher voltage is applied to the plate, the characteristic curve is not changed in shape, but is shifted toward the left. This is because larger negative grid voltages are required to reduce the space current to a given value.

This method of controlling the space current has a number of advantageous features. In the first place, it requires the expenditure of exceedingly small amounts of energy. If the grid is negative with respect to the filament, no electrons strike it and consequently no current flows through the battery C, hence the only energy drawn from it is that required to charge the condenser formed by the grid and filament, which is negligible in most cases. If, however, the grid is positive with respect to the filament, a few electrons strike it and a current is drawn from C which then supplies energy to the tube. If, however, the grid wires are very fine, this current may be made quite small even though relatively large positive potentials are impressed on the grid. The battery B may be one of high voltage and the space current will therefore have large amounts of power associated with it. Accordingly, by the expenditure of small amounts of power in the grid circuit, large amounts of power in the plate circuit may be controlled, and the device constitutes a relay having a large energy ratio.

In the second place, the response of the plate current to changes in grid potential is exceedingly quick, almost instantaneous. If the time required for an electron to travel from the filament to the plate is computed by eq. (3) of chap. XIII, it is found that for an ordinary tube with moderate plate voltages it is of the order of one hundredth of a millionth of one second. This then is the order of the time lag to be expected. For this reason it may be regarded as a relay with no moving mechanical parts and is therefore without inertia in its action.

Again, there exists for a considerable range, a linear relation between grid potential and plate current so that the variations in plate current are faithful reproductions of the changes in grid potential and thus the device is a distortionless amplifier.

203. Experiment 44. *Static Characteristics of a Three-element Electron Tube.*—Connect the apparatus as shown in Fig. 145. Use for the filament battery A, a set of storage cells, furnishing

from 10 to 20 volts depending upon the size of the tube to be tested. Ascertain from the instructor the normal heating current for the filament, and be careful that this is not exceeded at any time during the test. If the filament is of the oxide coated type, it should be operated at a dull red heat, but if it is a tungsten wire, bring it up to about the same brightness as the ordinary vacuum incandescent lamp. *B* may be a battery of flash light cells giving 500 volts or a motor generator set. For *C* use a battery of flash light cells giving about 60 volts. Bring the filament up to normal temperature, and apply a plate voltage of ⅓ normal. Apply a sufficient negative voltage to the grid to reduce the plate current approximately to zero. Raise the grid volts by steps to zero and positive values and note the grid

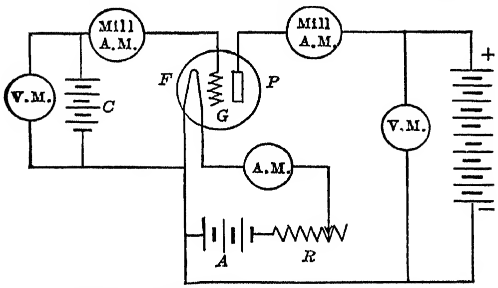

FIG. 145.—Three element electron tube.

and plate currents for each setting. Repeat for several values of plate voltage up to and including normal to obtain data for the family of curves shown in Fig. 147. Next hold grid voltage constant and measure the plate current for a series of plate potentials. Repeat for several different values of grid voltages, both positive and negative, to obtain data for the family of curves shown in Fig. 148.

Report.—Describe the three element electron tube and outline its principal operation features. Plot the static characteristic for the plate voltages studied, also the grid current as a function of grid volts. Sometimes a negative grid current is obtained. How can this be explained?

204. Amplification Factor.—The fact that the three-element tube may be used as a relay has been referred to several times, and it is necessary to define accurately what is meant by this statement. By a relay, is meant any device by which a small

amount of energy may be used to turn on and off or control a much larger source of energy. In the case of the electron tube, the source of energy is the plate battery and the grid is the gate by which it is controlled. Considering now the plate and grid circuits, it is obvious that we may be interested in the relative values of either the power, the currents, or the voltages existing in these circuits, and that we may accordingly refer to either the power amplification, the current amplification, or the voltage amplification. The meaning of the first two of these expressions is obvious; for example, by power amplification is meant the ratio of the change in power drawn from the plate battery to

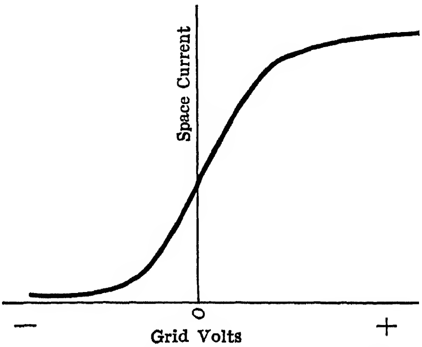

FIG. 146.—Characteristic for three element electron tube.

the change in power supplied to the grid, and a corresponding meaning is given to current amplification.

However, in the ordinary use of the tube, the voltage of the plate battery remains constant, and the meaning of the voltage amplification factor is not so evident. The significance of this term can perhaps be understood by reference to a series of static characteristics as represented in Fig. 147, where the dependence of space current upon grid volts for a series of plate potentials, at 50 volts intervals, is shown. Suppose, for example, the plate voltage is 100 and the grid volts zero. The space current is then 10 milliamperes. It is desired to increase the space current to 20 milliamperes. This may be done either by raising the plate voltage to 150 or the grid voltage to 5. Thus an increase of 5 volts on the grid produces the same change in the space current as an increase of 50 volts on the plate. The voltage amplification

factor in this case is said to be 10, since one volt on the grid is equivalent to 10 volts on the plate.

The relations illustrated in Fig. 147 may be shown by another set of static characteristic curves, Fig. 148 in which the grid

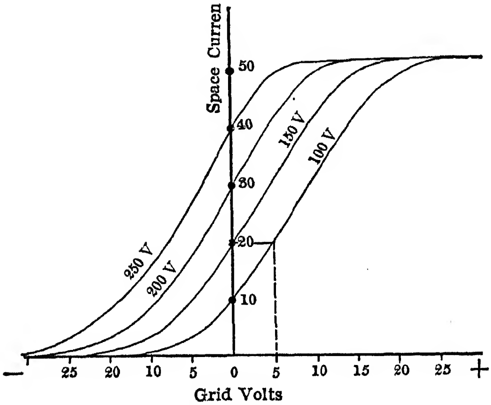

FIG. 147.—Dependence of static characteristics upon plate potential.

potential is kept constant and the plate potential is varied. The figure illustrates the dependence of plate current upon plate voltage for seven different grid voltages. This type of static characteristic is particularly useful in considering problems in modulation in radio telephony. By properly choosing the grid voltage it is usually possible to find a condition for which the plate current varies linearly with plate voltage over a considerable range.

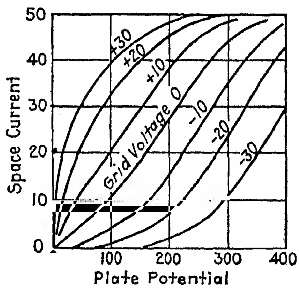

FIG. 148.—Dependence of space current upon plate potential for constant grid potential.

A working equation connecting these quantities may be deduced as follows. It was shown in eq. (2) that for the two-element tube, the plate current is proportional to the $\frac{3}{2}$ power of the plate voltage, i.e., $I_p = aV^{3/2}$, where a is a constant. Since a change in grid voltage is more effective by a certain factor, which we will call k, in producing a change in plate current than a change in the plate voltage, it follows that the plate current in a given tube on which there is acting a plate voltage E_p and a grid voltage E_g is just the same as though it

were a two-element tube with a plate voltage $E_p + kE_g$. The expression for the current then becomes

$$I_p = a(E_p + kE_g)^{3/2} \tag{4}$$

Since eq. (2) refers to the case in which there is an abundance of electrons at the filament and the current is limited only by the space charge, eq. (4) holds only for the left-hand part of the characteristic, i.e., up to the bend.

Referring to Fig. 147, it is seen that the static characteristics all have a point of inflection, and that for a considerable portion each side of this point, the curve is nearly a straight line. If the tube is used as a distortionless amplifier, it is necessary that the range of applied grid volts should not appreciably exceed the linear part. In this case, the simplified equation

$$I_p = a(E_p + kE_g) \tag{5}$$

may be used in which a is the filament to plate conductance of the tube, and is the slope of the linear part of the characteristic. If E_g is sufficiently negative, the plate current is zero. Calling this value E_{go} we have

$$k = -\frac{E_p}{E_{go}} \tag{6}$$

It is obvious that the amplification for a given tube depends upon the spacing of the grid wires. If these wires are far apart, a definite change of voltage is not as effective in controlling the electron flow as though the meshes were smaller. As a matter of fact, the amplification factor is inversely proportional to the distance between grid wires. Again, if the grid is close to the filament so that it acts upon the electrons before they have gained appreciable speeds, it is more effective than if it is near the plate. Thus, if it is desired to construct a tube with a large voltage amplification factor it should have a grid with a fine mesh mounted close to the filament. Tubes having amplification factors as large as 100 have been constructed, but in actual practice factors from 5 to 20 are more common.

A simple method for obtaining the amplification factor of a tube is to impress upon the plate a certain positive potential and then apply to the grid a negative potential sufficient to reduce the plate current to zero. The ratio of the plate and grid potentials is then the amplification factor of the tube for this particular plate voltage. It is found in practice that the amplification factor of a tube is not constant but varies with the plate and grid

potentials used. This is due to the fact that the average distribution of electrons between the plate and filament changes with the potentials on the grid and plate which in effect, changes their relative positions. By taking, in this manner, measurements over a series of values of plate voltage a fair idea of the behavior of the tube may be obtained.

While the method just described yields results sufficiently accurate for many purposes, it has nevertheless one serious error.

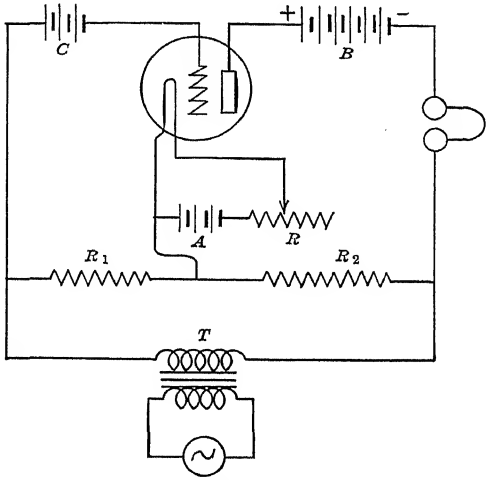

Fig. 149.—Dynamic method for amplification factor.

Unless the tube is very carefully designed, it does not have a sharp "cut off." That is, the characteristic curve does not proceed straight down to the axis, but slopes off and approaches it gradually. The actual negative grid potentials required to reduce the plate current to zero are much larger than would be obtained by continuing the straight portion of the characteristic until it intercepts the horizontal axis. In actual use, this intercept value is the one which is effective. A dynamic method in which this error is eliminated has been devised by Miller.[1] His circuit is shown in Fig. 149. The tube is connected in the ordinary way with a telephone receiver in the plate circuit, and potentials supplied to the plate and grid by the batteries *B* and *C* respectively. By properly adjusting the values of these voltages, the tube may be set at any point on the characteristic

[1] J. H. Miller, *Proc. Inst. of Radio Engineers*, vol. 12, 1918, p. 171.

curve. Included in the grid and plate circuits are the resistances R_1 and R_2 across which is connected the secondary of a telephone transformer T. When an alternating current is supplied to the primary of this transformer, small alternating voltages, i.e., the resistance drops across R_1 and R_2, are introduced into the grid and plate circuits respectively. It is obvious from the connections that when the additional voltage on the plate is positive that on the grid is negative and vice versa. By changing the relative values of R_1 and R_2 the ratio of these voltages may be made to have any desired value. If it is such that the added grid potential just balances that added to the plate, there will be no change in the steady plate current and consequently no sound in the phones. The amplification factor k is then the ratio of R_2 to R_1. The advantage of this method is that it measures the amplification factor while the tube is operating in the same manner as when actually used in practice. The dependence of the amplification factor upon the plate and grid volts may thus be easily and quickly obtained.

205. Experiment 45. *Amplification Factor of a Three-element Electron Tube.*—Connect the apparatus as shown in Fig. 149, using for P a pair of high resistance head receivers. The source B should furnish a voltage equal to the maximum for which the tube is designed, and if a power tube is under test, may be a high voltage generator. C should consist of a battery of small flash light cells. The A. C. supply should have a frequency high enough to give a good clear note in the phones, and the voltages across R_1 and R_2 should be low enough so that the operating point moves only a small amount along the static characteristic curve. Make two tests. First hold the grid volts at some predetermined value, and measure the amplification factor for a series of plate voltages ranging from a small value up to the maximum for which the tube is designed. Next hold plate volts at normal value and measure the amplification factor for a series of values of grid volts. Check the results of the first series by the static method explained above. That is, for each different plate voltage, find the negative grid potential required to reduce the plate current to zero.

Report.—Plot curves showing the dependence of the amplification factor upon both the plate and grid potentials by the dynamic method, and upon plate potentials for the static method. How do you account for the differences between these curves?

206. Internal Plate Resistance of a Three-element Electron Tube.—Following the amplification factor, the next most important characteristic of an electron tube from the standpoint of operation is perhaps its internal impedance. It is a well known principle of electrical practice that the impedance of a device should equal that of the circuit on which it operates. Accordingly, in designing a tube to operate on a particular circuit or conversely in adjusting a circuit to fit the tube which is supplying power to it, it is necessary to know the plate to filament impedance of the tube. The mechanism by which the vacuous space offers resistance may be understood by the following consideration. When a current flows through a conductor, heat is developed within it. This energy is furnished by the driving electric field which urges the electrons along through the conductor. Resistance, in this case, is due to a direct interference with the motion of electrons. As a consequence of this view of the nature of resistance, it might at first be thought that a perfect vacuum would be a perfect conductor of electricity since there is nothing to interfere with the free motion of electrons. That this however, is not the case is at once evident when one remembers that relatively large voltages are necessary to cause small currents to flow through the ordinary electron tubes, even when the conditions are far removed from those of current saturation. Moreover, the fact that it is easy to heat the plate red hot by the passage of current, indicates that it is accompanied by a consumption of energy.

When an electron is emitted by the heated filament, it finds itself in the electrostatic field existing between filament and plate, and it is at once accelerated toward the plate. Since the electron possesses mass, it necessarily gains kinetic energy as it moves toward the plate. This energy is abstracted from the electric field which accelerates it. When the electron strikes the plate, it possesses a velocity of the order of several thousand miles per second even under moderate potential differences. At the plate it is suddenly brought to rest and its kinetic energy of motion is converted into heat energy of the molecules of the plate. While the tube does not possess resistance in quite the same way that an ordinary metallic conductor does, it, nevertheless, consumes energy when a current passes, and it is customary to speak of its resistance and to define it on the basis of the energy it consumes. Thus, if I is the current flowing through

the tube, and W the watts consumed by it, its resistance R is defined to be such that

$$W = I^2R \tag{7}$$

Since this is the same equation as holds for the power converted into heat by the ordinary conductor, we may determine the resistance of the tube by the voltage required to furnish a given current through it. An application of Ohm's law to corresponding values of plate volts and plate current as read from the static characteristics shows that the resistance of a tube is not constant but depends upon the values of both the plate and grid potentials, and also upon the electron emission from the filament in case saturation voltages are used. It is necessary therefore to define the resistance of the tube for a particular point in the characteristic curve. This is done by saying that the resistance of the tube is the ratio of the change in plate volts to the change in plate current produced by it, when this change is made vanishingly small. That is

$$R = \frac{dE_p}{dI_p} \tag{8}$$

Thus the resistance is the reciprocal of the slope of the plate potential, plate current characteristic. Since this curve is seldom taken in practice, R may be obtained from the plate current-grid potential characteristic by remembering that

$$E_p = kE_g \tag{9}$$

whence

$$dE_p = kdE_g \tag{10}$$

Therefore

$$R = k\frac{dE_g}{dI_p} \tag{11}$$

The internal plate resistance is then the product of the amplification factor and the reciprocal of the slope of the plate current-grid potential characteristic.

While this method is satisfactory for many purposes, it is open to the objection that it requires a determination of the amplification factor k. A dynamic null method has been employed by Ballantine[1] in which the resistances may be measured directly. The connections for this circuit are shown in Fig. 150. It will be noted that the arrangement is essentially a Wheatstone bridge in which the plate to filament path through the tube is one

[1] BALLANTINE, *Proc. Inst. Radio Engineers*, vol. 7, 1919, p. 129.

of the arms. Because of the battery B a steady current flows through all four arms of the bridge and also through the phones. The phones, however, respond only to the variable currents furnished by the A.C. supply. The resistance thus measured will be those defined by eq. (8).

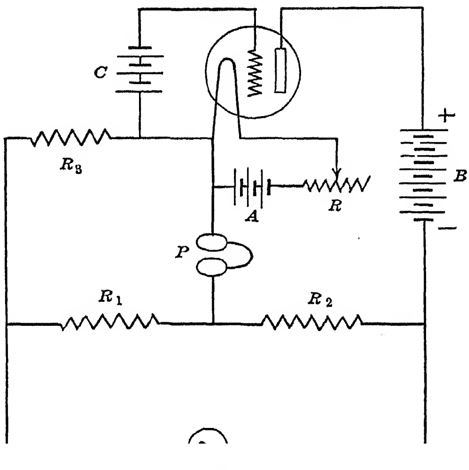

FIG. 150.—Connections for measuring resistance of tube.

207. Experiment 46. *Plate-filament Resistance of an Electron Tube.*—Connect the apparatus as shown in Fig. 150. As a source of alternating voltage use any oscillator giving a good clear note furnishing an E.M.F. of about 10 volts. The resistance R_3 should be of the same order of magnitude as the tube under test, i.e., several thousand ohms. Ascertain the normal plate voltage for the tube, and make a series of measurements of internal resistance varying the grid volts over a considerable range, both positive and negative. Repeat using plate voltages three-quarters, one-half and one-quarter normal. Disconnect the tube from the bridge and determine its static characteristic for normal plate voltage.

Report.—Plot internal resistance as a function of grid volts for the four series of observations. Plot the static characteristic and check your results by the first method described. The amplification factor may be obtained by extending the straight portion of the characteristic and taking its intercept on the

horizontal axis as the value of the grid volts necessary to reduce the plate current to zero. See Art. **204.**

208. Use of Two Element Tube for High Voltage D.C. Supply. The rectifying property of the two-element tube may be taken advantage of to maintain from an A.C. supply a charge in a condenser and this charge, when slowly drained to supply current to the plates of other tubes, replaces the B battery. Such an arrangement, when equipped with an appropriate electrical filter for smoothing out the ripples due to the intermittent charges given to the condenser, is often called a "B Battery Eliminator," and is very generally used to furnish the plate voltage for receiving sets, power amplifiers, radio transmitters, X-ray tubes, etc.

When used in this manner, the two-element tube, or kenotron, has three important advantages: It rectifies the highest voltages

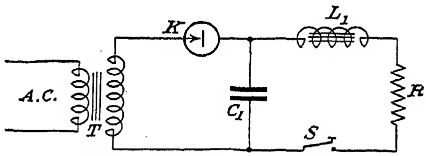

FIG. 151.—Simple rectifier circuit.

met in practice if pumped to a sufficiently high vacuum; the rectifying property is independent of the frequency of the supply; no reverse current flows through the rectifying tube unless the supply frequency is so high that the inter-element capacitance of the tube offers an appreciable admittance. To illustrate the action of such a circuit, suppose T of Fig. 151 is a transformer whose primary is connected to an A.C. source of low frequency, e.g., 60 cycles. K is some device, e.g., a two-element tube, which permits current to flow in the direction of the arrow but not in the reverse direction. If the switch S is open, during each half of the cycle in which the voltage of the secondary is such as to cause a current in the direction of the arrow, the condenser receives a charge as indicated. Since this charge cannot escape backward through the transformer because of K, the condenser receives increments of charge until its voltage builds up to the peak value of the wave impressed.

If the switch S is closed, a portion of the charge in C_1 leaks out through R, thus supplying a direct current diminishing in value,

since the voltage across C_1 decreases with its charge. When the voltage of T is again in the positive direction and has risen to a point slightly in excess of that of C_1, as at B in Fig. 152, the condenser again receives a charge bringing its voltage very nearly back to the original peak value. R is thus supplied with a direct current, having a slight ripple of frequency equal to that of the A.C. supply. The larger R, the smaller the amplitude of the ripple. If R is not too large, an inductance L_1 inserted in series will appreciably reduce the amount of ripple.

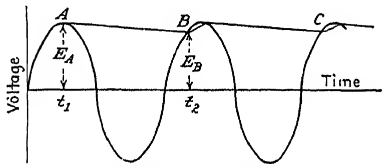

FIG. 152.—Voltage from half-wave rectifier.

With $L_1 = O$, the fluctuation of voltage δE_1 across R may be computed as follows: Calling E_1 the voltage across C_1, the current through R is given by

$$i = -C_1\frac{dE_1}{dt} \quad \text{or} \quad idt = -C_1 dE \tag{12}$$

Integrating both sides and putting

$$t_2 - t_1 = t \qquad \text{and} \qquad E_A - E_B = \delta E_1,$$

$$it = C_1\delta E_1 \tag{13}$$

Since t is nearly one complete cycle of the A.C. supply, we have

$$t = \frac{2\pi}{\omega} \qquad \text{and} \qquad i = \frac{E_1}{R}$$

where E_1 is approximately the peak value of the A.C. supply. Substituting in eq. 13, we have

$$\frac{E_1}{R}\frac{2\pi}{\omega} = C_1\delta E_1$$

or

$$\frac{\delta E_1}{E_1} = \frac{2\pi}{C_1\omega R} \tag{14}$$

Equation 14 gives the percentage fluctuation of voltage across R and is seen to be smaller the larger the value of R_1, the higher the frequency and the larger C_1. If the inductance L_1 is

included, the fluctuation δE_r across R is such a fraction of δE_1 as R is of the total impedance across C_1. Thus

$$\frac{\delta E_r}{\delta E_1} = \frac{R}{\sqrt{R^2 + L^2\omega^2}} \tag{15}$$

Multiplying (14) by (15), we have

$$\frac{\delta E_r}{E_1} = \frac{2\pi R}{C_1\omega\sqrt{R^2 + L^2\omega^2}} \tag{16}$$

If R is small enough so that $L^2\omega^2$ is large compared to R^2, an appreciable reduction in the fluctuation results. A much more

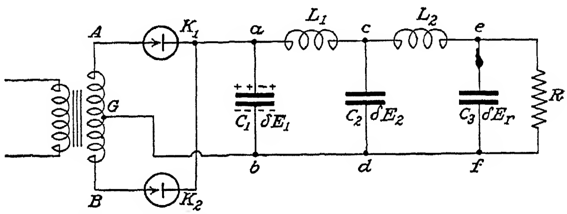

FIG. 153.—Double-wave rectifier with filter.

effective arrangement is afforded by the circuit shown in Fig. 153 where both halves of the wave are used to charge the condenser and the ripple is smoothed out by an extended filter system. It is obvious that when A is positive, a current flows through K_1 charging C_1 as indicated and that any charge already in C_1 is prevented by K_2 from leaking out through the transformer. Similarly when B is positive, a charge flows to C_1 in the same direction as before and its leakage is prevented by K_1. The frequency of charging the condenser is thus twice that of the A.C. supply and by eq. 14 the ripple is reduced by a factor 2. This circuit, however, has the disadvantage that the maximum voltage obtainable across C_1 is only half the peak value across the terminals of the transformer while with the single rectifier system the total voltage is secured. The voltage fluctuations across C_1 are shown in Fig. 154.

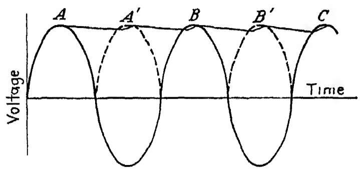

FIG 154.—Voltage from double-wave rectifier.

The effectiveness of the filter system in reducing the ripple across R may be computed as follows:

Let δE_1, δE_2, δE_r = voltage fluctuations across C_1, C_2 R
Z_3 = impedance between e and f
$Z_3' = jL_2\omega + Z_3$ = impedance shunted across C_2
Z_2 = impedance between c and d
$Z_2' = jL_1\omega + Z_2$ = impedance shunted across C_1

Then

$$\frac{\delta E_r}{\delta E_2} = \frac{Z_3}{Z_3'} \qquad \frac{\delta E_2}{\delta E_1} = \frac{Z_2}{Z_2'} \qquad \frac{\delta E_1}{E_r} = \frac{2\pi}{C_1\omega R} \tag{17}$$

It is assumed that there is no resistance drop through L_1 and L_2 so that the steady voltage across R is the same as across C_1. Multiplying together the three equations (17), we have

$$\frac{\delta E_r}{E_r} = \frac{Z_3}{Z_3'} \cdot \frac{Z_2}{Z_2'} \frac{2\pi}{C_1\omega R} \tag{18}$$

Expressing the impedances in the manner of the complex algebra, we have

$$\frac{1}{Z_3} = jC_3\omega + \frac{1}{R} = \frac{1 + jC_3\omega R}{R}$$

$$Z_3 = \frac{R}{1 + jC_3\omega R} \tag{19}$$

$$Z_3' = jL_2\omega + \frac{R}{1 + jC_3\omega R} = \frac{R(1 - L_2C_3\omega^2) + jL_2\omega}{1 + jC_3\omega R} \tag{20}$$

$$\frac{1}{Z_2} = jC_2\omega + \frac{1}{Z_3'} = jC_2\omega + \frac{1 + jC_3\omega R}{R(1 - L_2C_3\omega^2) + jL_2\omega}$$

$$Z_2 = \frac{R(1 - L_2C_3\omega^2) + jL_2\omega}{1 - L_2C_2\omega^2 + jR\omega(C_2 + C_3 - L_2C_2C_3\omega^2)} \tag{21}$$

$$Z_2' = jL_1\omega + \frac{R(1 - L_2C_3\omega^2) + jL_2\omega}{1 - L_2C_2\omega^2 + jR\omega(C_2 + C_3 - L_2C_2C_3\omega^2)} \tag{22}$$

or

$$Z_2' = \frac{R(1 - L_2C_3\omega^2) - RL_1\omega^2(C_2 + C_3 - L_2C_2C_3\omega^2) + j(L_1\omega + L_2\omega - L_1L_2C_2\omega^3)}{1 - L_2C_2\omega^2 + jR\omega(C_2 + C_3 - L_2C_2C_3\omega^2)} \tag{23}$$

Substituting equations 19, 20, 21 and 22 in 18 and remembering that the absolute value of a vector is the square root of the sum of the squares of its real and imaginary components, we have

$$\frac{\delta E_r}{E_r} = \frac{2\pi}{C_1\omega\{[R(1 - L_2C_3\omega^2) - RL_1\omega^2(C_2 + C_3 - L_2C_2C_3\omega^2)]^2 + [L_1\omega + L_2\omega - L_1L_2C_2\omega^3]^2\}^{1/2}} \tag{24}$$

209. Experiment 47. *Efficiency of a B Battery Eliminator.*—Set up the apparatus as shown in Fig. 155 where K_1 and K_2 are the rectifier tubes supplied with power from the transformer T. The condensers C_1, C_2, C_3 may be of any convenient size, e.g.,

two to four microfarads. The inductances L_1 and L_2 should be of the order of 30 henries. The input power is measured by the watt meter, W.M., and the output by taking the product of the D.C. voltage and current measured by V.M. and M.A. respectively. R should be a variable high resistance capable of dissipating 50 watts or more. The filaments of the rectifier tubes may be heated by a storage battery, as shown in the figure, or, as is usually the case, by alternating current from a special winding on the transformer T.

Measure first the core loss in the transformer. To do this, disconnect the filter at G and D, the filament heating circuit, and the plate circuit of the rectifiers. Apply normal voltage to the transformer primary and read the watt meter. If the filaments are heated by the transformer, connect this circuit and note the

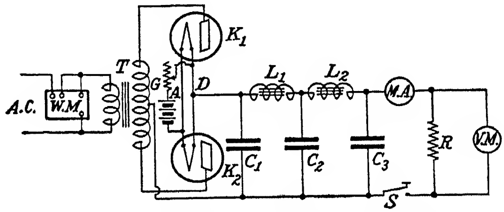

FIG. 155.—Circuit for B battery eliminator.

additional watts when power is supplied to the primary. If the filaments are separately heated, measure the filament power consumption by voltmeter and ammeter method. Next connect the filter at G and D and with S open, measure the total no-load input to the system. Make an efficiency run varying R so as to change the D.C. output current through as wide a range as possible. Read input watts and output voltage and current.

Report.—Record your findings regarding losses in the transformer core, filaments of rectifier tubes and no load-filter loss. Plot (a) efficiency and (b) output D.C. voltage as a function of output current. State your findings regarding the ripple across the various condensers. Substitute the values of the condensers and inductances in eq. 24 and compute the per cent fluctuation in output voltage for some value of R used in the test.

210. The Tungar Rectifier.—In the case of tubes operated on a pure electron discharge, it is possible, at best, to obtain currents of but a fraction of an ampere, and these only by the employment of several hundred volts. While such tubes are

satisfactory for the rectification of high voltage currents they are, nevertheless, unsuitable for cases in which several amperes at low voltage are required, as, for example, charging storage batteries from ordinary city lighting circuits. For this purpose, a satisfactory tube, known as the tungar rectifier has been developed by the General Electric Co.[1] It is a two-element tube, the cathode of which is a heated tungsten filament in the form of a helix, while the anode is a conical piece of tungsten mounted about 3 mm. from the filament. Instead of a vacuum, the tube contains pure argon at a pressure of 8 or 10 cms. of mercury.

The purpose of the argon is to furnish positive ions which neutralize the space charge encountered in pure electron tubes, and thus to reduce by many fold the voltage required to maintain the current. Furthermore the positive ions take part in transporting electricity between the electrodes and thus materially increase the carrying capacity of the tube. In the early attempts to utilize positive ions, it was found that many gases have injurious effects. For example, in the presence of oxygen, the electron emission of tungsten is cut down to a small fraction of what it is in high vacuum. Again, many gases unite with the heated filament forming compounds, which are highly volatile at normal operating temperatures and thus cause it to disintegrate. Furthermore, when a gas is present in only small amounts, the mean free path of the positive ions may be so great that they acquire velocities sufficient to chip off particles of the filament softened by heating, and thus hasten its disintegration. By use of an inert gas such as argon, the first two difficulties are overcome and by shortening the mean free path by using relatively high pressures, the speeds are so reduced by frequent collisions that the disintegration by bombardment is insignificant.

In order to avoid the formation of volatile compounds it is necessary that the argon be very pure, and in the early tubes great pains were taken to secure this. It has been found possible to mount within the tube, usually on one of the filament leads, substances which react chemically with the impurities, which thus keep the argon in a pure state. For the larger sized tubes, a graphite anode mounted on a tungsten support is often used, and the purifying agent may then be introduced in the anode. As impurities are given off from the electrodes or interior walls,

[1] *Gen. Elec. Rev.*, vol. 19, No. 4, 1916, p. 197.

the drop across the arc increases, liberating more heat at the anode, which thus causes vapors to be given off by the purifying agent and in this way the argon is maintained in a state of high purity.

After the arc has once been started, the filament may be kept heated by positive ion bombardment after the heating current has been shut off. In this case, the arc confines itself to a very limited portion of the filament. This spot wastes away more rapidly than the rest of the filament and the life of the tube is materially shortened when operated in this way. For the larger sized tubes, i.e., those with a current capacity of 20 to 40 amperes, a fine tungsten point is independently mounted close to the filament. This may be heated to a high temperature by using it as anode with the filament as cathode. If the connections are then shifted, this hot point may be used as the cathode against the regular anode, its temperature being maintained by positive ion bombardment as just explained. The filament serves then as a starting device only and the tube has an exceedingly long life. Since relatively large amounts of power are consumed by the filament current, it might be expected that the latter method of operation would result in a material increase in efficiency. This is not the case, since the voltage across the arc rises when the filament current is cut off, and the resulting increase in energy consumption in the arc itself practically balances the saving effected in the filament. Commercial sets are usually made for the purpose of charging automobile storage batteries with a maximum E.M.F. of 60 volts directly from 110 volt alternating current circuits. To avoid losses in controlling rheostats, a step down transformer is mounted within the case to reduce the A.C. voltage to the desired value before rectifying it. A separate low voltage winding is included for heating the filament.

211. Experiment 48. *Study of the Tungar Rectifier.*—For simplicity of operation obtain the characteristic curves by the use of direct current. Mount the tube in a special socket and connect it as shown in Fig. 156. Ascertain the normal heating and load current and be careful not to exceed these values. Determine the voltage drop across the arc for different load currents.

Next place the tube in the socket of the regular rectifier set and make an efficiency run using the 110 volt A. C. circuit as a source of power. Measure the input by means of a wattmeter and the

output by the volt-ampere product for the load rheostat *R* (Fig. 157). Record the load current and load drop as determined by both A. C. and D. C. instruments. Vary the load current

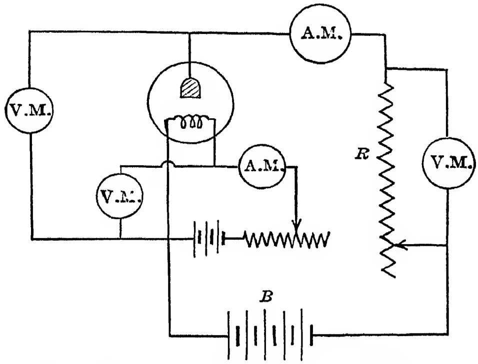

FIG. 156.—Connections for tungar rectifier.

through as wide a range as the tube will permit. Before starting the test open up the housing for the set and study carefully the internal connections.

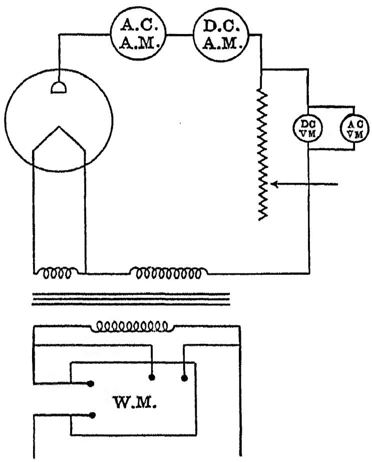

FIG. 157.—Connections for rectifier mounted in commercial set.

Report.—Plot a curve of the voltage drop across the arc against the load current for the tungar bulb. Discuss the shape of the curve. Also plot a curve of the efficiency against the load cur-

rent when the rectifier is used on an A. C. Compare the readings of the A. C. and D. C. ammeters and voltmeters. Compute the ratio of the average value of the sine wave rectified current to the root-mean-square value and compare this ratio to the observed ratio of D. C. to A. C.

CHAPTER XVII

OSCILLATORS AND AMPLIFIERS

212. The Vacuum Tube Oscillator.—In article 121 page 159, a method was described by which a three-element electron tube may be used to maintain continuous oscillations in a circuit containing resistance, inductance, and capacitance, and a qualitative explanation was given of the manner in which the tube supplies to the oscillating circuit the energy necessary for this purpose.

A somewhat more quantitative discussion of the process will now be given. Suppose we have an ordinary parallel resonance circuit supplied with power from a battery E through a device in series with it, as indicated by the rectangle in Fig. 158. For the present we will not describe this device further than to say it has a resistance R_1. Let us seek the property which the device must have in order that it may maintain the circuit LCR in continuous oscillation. Writing Kirchhoff's laws for this network we have

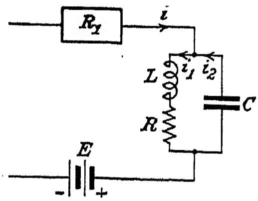

FIG. 158.—A symbolic circuit driver.

$$i = i_1 - i_2 \tag{1}$$

$$R_1 i - \frac{1}{C}\int i_2 dt = E \tag{2}$$

$$L\frac{di_1}{dt} + Ri_1 + \frac{1}{C}\int i_2 dt = 0 \tag{3}$$

Eliminating i between (1) and (2), we have

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{1}{R_1C}i_2 \tag{4}$$

Differentiating (3), substituting (4), and collecting terms we have for the current through the condenser

$$\frac{d^2i_2}{dt} + \left(\frac{1}{R_1C} + \frac{R}{L}\right)\frac{di_2}{dt} + \left(\frac{R}{R_1} + 1\right)\frac{1}{LC}i_2 = 0 \tag{5}$$

This equation is of the same form as equation 15, article 103, for a simple circuit containing resistance, inductance and capac-

itance, in which the condenser had been charged and was allowed to discharge through the resistance and inductance. It was there shown that if $R^2C^2 < 4LC$, the solution of equation 15 is

$$i = I_e^{-\alpha t} \sin \omega t \tag{6}$$

in which

$$I = \frac{2Q}{\sqrt{4LC - R^2C^2}}, \quad \alpha = \frac{R}{2L}, \qquad \frac{\sqrt{4LC - R^2C^2}}{2LC}$$

The form of this solution is the damped sine wave of Fig. 72. The coefficient α is called the damping factor and determines the rate at which oscillations die out. It is proportional to the resistance of the circuit; hence the larger the resistance, the more rapidly the oscillations are damped. If α is zero, oscillations, once started, continue indefinitely; and if α is negative, the amplitude increases. Moreover it is to be noted that α is half the coefficient of $\frac{di}{dt}$ in equation 15 above.

Applying the same reasoning to equation 5 for the circuit of Fig. 158, it may be seen that its solution represents also a damped sinusoidal current having a damping factor given by the coefficient of the first order derivative,

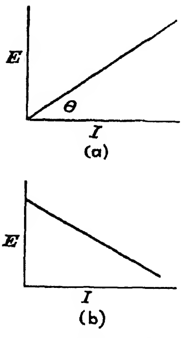

FIG. 159.—Positive and negative resistance.

$$\alpha = \frac{1}{2}\left(\frac{1}{R_1C} + \frac{R}{L}\right). \tag{7}$$

The condition then that oscillations, once started in this circuit, may persist is that $\alpha = 0$, or

$$R_1 = -\frac{L}{RC} \tag{8}$$

In other words, if the device has a "negative resistance" equal to the impedance of the oscillatory circuit with which it is joined in series, it will maintain any oscillations that may be started in this circuit. Moreover, if it has a negative resistance greater than this value, it will cause oscillations, once started, to build up.

We must now define the meaning of the term "negative resistance." This may be done by noting that for a positive resistance, that is, one obeying Ohm's law, the voltage across it is directly proportional to the current through it and that the resistance is the slope of the straight line as in (a) Fig. 159.

$$E = RI; \quad \tan \theta = R \tag{9}$$

Conversely, a circuit having a negative resistance is one in which the slope of the EI curve is negative, as in (b) Fig. 159. Accordingly, any device so constituted that the voltage across it decreases as the current through it increases may be said to possess negative resistance and, when properly connected in a circuit, may be used to convert energy from a D.C. source into energy of electrical oscillations. The carbon arc and the tungar rectifier are devices of this sort.

A three-element electron tube, because of its amplifying property, may, if properly connected into the circuit, be made to function as a negative resistance. Such a connection is shown in Fig. 160. R_1 and R_2 are joined in series across E, a source of alternating E.M.F. It is to be noted that a current through them in the arrow direction, gives to the plate a positive potential and to the grid a negative potential whose relative values depend upon the ratio of R_1 to R_2. Since the plate current is given by

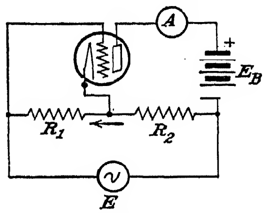

FIG. 160.—Vacuum tube connected for negative resistance.

$$i_p = \frac{1}{R_p}(e_p + ke_g), \tag{10}$$

it is obvious that for an increase Δe_p in plate potential, the plate current may increase, remain constant or decrease, according as Δe_g is less than, equal to, or greater than $k\Delta e_p$. Thus the effective resistance between the plate and filament elements of the tube, as defined by $\frac{\Delta e_p}{\Delta i_p}$, may be either positive, zero or negative, depending upon the changes in grid potential simultaneously applied. An alternating E.M.F. at E thus produces through the tube an alternating current which may be in phase with the voltage at the plate, i.e., tube resistance positive, or a current opposite in phase to the plate voltage, tube resistance negative. Thus by accompanying all voltage changes on the plate by an opposite voltage change on the grid supplied from another part of the circuit, the tube may be caused to function as a negative resistance and hence can maintain oscillations in a circuit.

Of the many ways in which this may be accomplished, one of the simplest is that shown in Fig. 161 in which LCR is the oscillatory circuit and E_B the source of D.C. power. C_1 is a blocking condenser to prevent the upper part of L from short-circuiting

the tube and to transfer the voltage across the plate turns of L to the plate. It should be large compared to C. L' is a choke coil to prevent the alternating voltage between plate and filament from being short-circuited by the battery. The filament, which is supposed to be grounded, is connected to a point somewhat below the middle of L. E_c is a battery to maintain the average potential of the grid at any desired point on the static characteristic of the tube.

Suppose that by any change of conditions, e.g., closing the plate circuit, oscillations are set up in LCR and that the current at a given instant is flowing in the arrow direction and increasing. Because of the inductance in the upper part of L, a positive voltage is supplied to the plate, while at the same time the voltage drop across the lower part of L supplies a negative voltage to the grid. As the oscillations continue, grid and plate are supplied

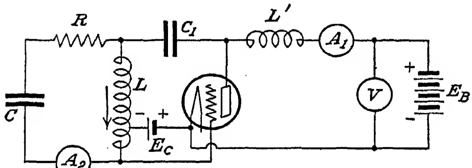

FIG. 161.—Vacuum tube oscillator.

with voltages opposite in phase, the condition necessary for the tube to function as a negative resistance. If the ratio of the turns of L across grid and plate has a suitable value, oscillations build up until the positive resistance of the tube, due to the approach to the saturation limit of the plate current, offsets its negative resistance.

213. Efficiency of a Vacuum-tube Oscillator.—One of the important considerations concerning such an oscillator is its efficiency. The input to the system, neglecting filament power, is the product of the D.C. volts supplied by the battery E_B and the average value of the current as measured by the D.C. ammeter A_1. If all the energy of the oscillations is used in heating the resistance R, the output is I^2R where I is the effective value of the current measured by A_2. The difference between the input and the output appears mainly as heat developed at the plate by impact of the electrons as they arrive from the filament.

In attempting to set definite limits to the efficiency that may be secured, two special cases will be considered depending upon

the operating point selected on the static characteristic. Neither of these can be actually realized in practice, but may be closely approached.

(1) **Tube Operated at Mid Point of Static Characteristic.**—We will assume here the ideal condition in which the peak value of the A.C. voltage applied to the plate is equal to the steady voltage of the battery as represented by (*a*) in Fig. 162. We assume also that the peak value of the A.C. plate current equals the D.C. plate current. The plate current is in phase with the grid volts, and hence is 180° out of phase with the plate volts as shown in (*b*). Calling e and i the instantaneous voltage and current in the plate circuit, we have

$$e = E + E \sin \omega t \text{ and } i = I - I \sin \omega t \quad (11)$$

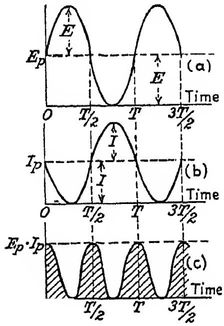

FIG. 162.—Current and voltage relations for tube operated at mid-point of static characteristic.

The power supplied by the battery, i.e., the input is

$$W_i = \frac{1}{T}\int_0^T Eidt = \frac{1}{T}\int_0^T E(I - I\sin\omega t)dt = \frac{EI}{T}\int_0^T (1 - \sin\omega t)dt = EI \tag{12}$$

T is the time of one complete oscillation. The power supplied to the tube is

$$\begin{aligned} W_t &= \frac{1}{T}\int_0^T eidt = \frac{1}{T}\int_0^T (E + E\sin\omega t)(I - I\sin\omega t)dt \\ &= \frac{EI}{T}\int_0^T (1 - \sin^2\omega t)dt = \frac{1}{2}EI \end{aligned} \tag{13}$$

Since the power supplied to the tube is the only loss, we have for the efficiency,

$$\text{Efficiency} = \frac{W_i - W_t}{W_i} = 50\%$$

The product of plate volts and plate current is shown in (*c*) Fig. 162. The area under this curve, shown shaded, represents the energy supplied to the tube in a given time, while that above the curve, unshaded, represents that supplied to the oscillatory circuit.

(2) **Tube Operated at "Cut-off" Point on Static Characteristic.**—We will assume now another set of ideal conditions in which the tube is "biased" so that no plate current flows except

during the positive half of the grid-voltage cycle. Assuming the plate voltage to be as in the previous case, the plate voltage and current relations are as shown in (*a*) and (*b*) respectively of Fig. 163. The expressions for e and i given in equation (11) hold for this case also except that the steady current term I is zero. The integrations for power are to be carried out only for that portion of the cycle in which current flows; that is from $\frac{T}{2}$ to T. Accordingly, the power furnished by the battery is

$$W_i = \frac{1}{T}\int_{T/2}^{T}(-EI\sin\omega t)dt = EI$$

$$.318EI \quad (13)$$

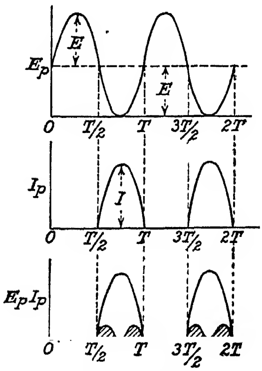

FIG. 163.—Voltage and current relations for tube operated at cut-off point on static characteristic.

The power supplied to the tube, or the loss in heating the plate, is

$$W_t = \frac{1}{T}\int_{T/2}^{T} - (E + E\sin\omega t)I\sin\omega t dt =$$

$$-\frac{EI}{T}\left[\int_{T/2}^{T}\sin\omega t dt + \int_{T/2}^{T}\sin^2\omega t dt\right] =$$

$$EI\left(\frac{1}{\pi} - \frac{1}{4}\right) = .068EI \quad (14)$$

The efficiency is then

$$\text{Efficiency} = \frac{W_i - W_t}{W_i} = 78.6\ \%$$

The energy supplied to the tube, given by the product of curves (*a*) and (*b*) of Fig. 163, is shown by the shaded area under the double humped curve; that furnished by the battery, by the sine curve; while that supplied to the circuit is the difference between these, i.e., the unshaded area.

214. Experiment 49. *Efficiency of a Vacuum Tube Driven Oscillator.*—Connect the apparatus as shown in Fig. 164 which is the same as in Fig. 161 except that another method of securing the grid bias is indicated. Instead of the battery E_c, a condenser C_2 is connected between one end of the coil L and the grid. This condenser must be large enough to transfer the voltage to the grid without appreciable loss. If the voltmeter V_2 were disconnected, a considerable difference of potential would build up across this condenser because of the unilateral conductivity of the grid-filament elements of the tube. That is, each time the

grid becomes positive, because of the oscillations in LCR, it conducts, giving a charge to C_2 as indicated. This produces a negative potential on the grid since the positive plate of C_2 is connected to the filament and is, therefore, at zero potential. This negative grid potential, after a few oscillations, builds up to the point where the plate current is reduced to zero, and the oscillator stops. If, however, a high resistance is shunted across C_2, the accumulated charge slowly leaks away, and by properly choosing this leakage resistance, any desired grid bias may be secured. For low-powered oscillators, a D.C. voltmeter may be used as the leak resistance, and serves the additional purpose of measuring directly the grid bias. R is an inductance and capac-

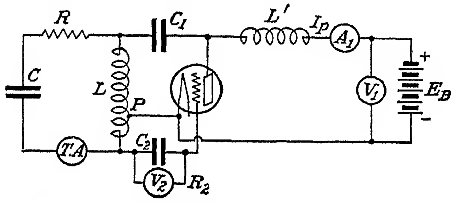

FIG. 164.—Connections for measuring efficiency of an oscillator.

itance free resistance box with tenth ohm steps. To measure the resistance of the coil and thermo-ammeter combined, disconnect LCR and $T.A.$ from the circuit and measure their combined resistance as in experiment 38, inserting a thermo-galvanometer in series. The resistance of the thermo-galvanometer must, of course, be subtracted from the results thus secured.

Choose a suitable load resistance R and adjust the ratio of grid and plate turns so that the reading of the thermo-ammeter $T.A.$ is a maximum. With a constant supply voltage E_B, take a series of readings varying R from zero to the largest value for which readable indications on $T.A.$ may be secured, recording input voltage and current, output radio frequency current and grid bias.

Compute the output power I^2R for this series. Set R at the value for which the output is greatest and, keeping it at this constant value, take a second series of observations, varying the supply voltage E_B through a wide range, recording the same quantities as above. Measure the filament voltage and current, and compute the power consumed by it. Record R_2, the resistance of the voltmeter, V_2.

Report.—Compute for both series of observations watts output, $I_A{}^2R$; watts input to plate circuit, V_1I_p; and grid circuit loss, $V_2{}^2/R_2$. Compute the efficiency, i.e., watts output divided by plate circuit input for each series. Plot, for the first series, efficiency as a function of load resistance, and for the second, efficiency against plate-circuit input. Plot grid circuit loss for each series. Explain the action of the tube in maintaining continuous oscillations.

215. Radio Frequency Amplifier.—Experiment 49 illustrated the use of the three-element electron tube as a generator of continuous oscillations. For this purpose, a portion of the voltage across the inductance of the oscillating circuit was used to supply the A.C. voltage to the grid and the proper phase relation, i.e., 180° between grid volts and plate volts was secured by using

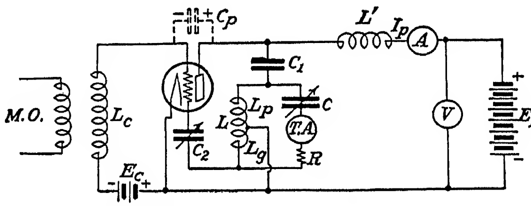

FIG. 165.—Radio frequency amplifier.

the potential drop across the remainder of this coil to supply the plate voltage.

In many cases it is advantageous to supply the A.C. grid voltage, not from the oscillatory circuit itself, but from an independent oscillator. This is common practice in radio transmitters, where the exciting circuit is a master oscillator, usually stabilized by a quartz piezo-electric crystal. The tube then functions as a radio-frequency amplifier. Such a circuit is shown in Fig. 165, in which LCR is the oscillatory circuit as before, connected as a parallel resonance circuit between plate and filament with the condenser C_1 in series to prevent the plate turns of the coil L from short-circuiting the tube. C_1 should be somewhat larger than C. The grid excitation is supplied by the voltage induced in the coupling coil L_c from the master oscillator, M.O., and the circuit LCR is tuned to the same frequency as $M.O.$ The choke coil L' serves the same purpose as in the previous case, i.e., it prevents the battery E_B from short-circuiting the radio-frequency voltage between plate and filament.

An amplifier circuit such as this has a tendency to oscillate by itself independent of the master oscillator. The reason for this is that energy is fed back from the plate circuit to the grid through the capacity coupling between the grid and plate, which constitute a condenser C_p, indicated by the dotted lines in the figure. During the positive half of the cycle of oscillation in LCR, the plate is at a higher potential than the grid, resulting in an accumulation of charge on this condenser as indicated by the + and − signs. During the other half of the cycle, the charges are reversed. Thus, because of this internal tube capacitance, the grid receives a voltage always opposite in phase to that on the plate, which, as pointed out above, is the condition necessary for self-oscillations.

To prevent self-oscillations it is necessary to supply to the grid, through an auxiliary coupling circuit, a voltage equal and opposite to that furnished through C_p, and several methods for accomplishing this have been devised. One of the simplest is that shown in the figure where C_2 is connected between the grid and the end of the coil L opposite the plate connection. Prevention of self-oscillations in a radio-frequency amplifier is termed "neutralization" and C_2 is called the neutralizing condenser. The condition for neutralization may be obtained as follows: The circuit $L_gC_2C_pL_p$ is equivalent to a Wheatstone bridge as shown in Fig. 166. The oscillatory current through LCR produces a voltage across L corresponding to E in this figure. It is desired to keep the grid at the same potential as the filament. Calling i_2 the instantaneous current through the coils L_g and L_p and i_1 that through C_2 and C_p, we have, for the balance condition of this bridge

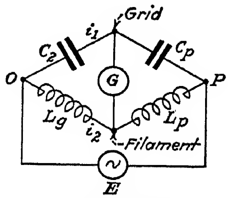

FIG. 166.—Equivalent bridge circuit for neutralization.

$$L_g\frac{di_2}{dt} = \frac{1}{C_2}\int i_1dt$$

$$L_p\frac{di_2}{dt} = \frac{1}{C_p}\int i_1dt$$

Dividing one equation by the other, we have

$$\frac{L_g}{L_p} = \frac{C_p}{C_2} \text{ or } L_gC_2 = L_pC_p \tag{15}$$

When this condition is satisfied, the potential of the grid is unaffected by voltage changes on the plate, and variations in

plate current depend only on the voltage supplied to the grid by the coupling coil L_C.

216. Experiment 50. *Efficiency of a Vacuum-tube Amplifier.*—Connect the apparatus as shown in Fig. 165 using for the master oscillator any suitable circuit driver such as that of Exp. 38. C_2 should be a small variable condenser of only a few micro-microfarads. Remove the circuit LC and obtain the resonance setting by placing it directly in inductive relation with $M.O.$ Measure its resistance, including that of the thermo-ammeter $T.A.$ by the method of Exp. 38. The resistance R of Fig. 165 is the variable loading resistance and is not to be included in this measurement.

Return the circuit LC to its place in the amplifier and adjust C_2 for neutralization. The method for doing this may be understood by referring again to Fig. 166. It is well known that for any balanced bridge, the galvanometer and power supply may be interchanged without disturbing the balance. Accordingly, if the condition of eq. 15 is satisfied with power supplied at P and O, then P and O will be at the same potential when power is supplied at the points marked Grid and Filament. Thus in Fig. 165 the ends of the coil L will be at the same potential when a voltage is applied across the grid and filament if condition 15 is satisfied. The thermo-ammeter $T.A.$ may serve as the galvanometer without change of connection. Disconnect the filament battery, supply voltage to the grid and filament from $M.O.$ through L_2, and, with LC tuned to resonance with $M.O.$ adjust C_2 until no current flows through $T.A.$ The balance condition is then secured and the amplifier is "neutralized." A more sensitive indicator than the thermo-ammeter may be secured by connecting a single loop of wire to a thermo-galvanometer and placing it near the coil L.

With the circuit neutralized, take two efficiency runs as in Exp. 49. First hold E_B constant and vary the load resistance, measuring the D.C. plate voltage and current, and the radio-frequency current in LC. Second, using the value of R for which the output power is a maximum, make a second run varying E_B through a wide range. In each case take the product of V and I_p as the power input, and the square of the oscillating current multiplied by the total resistance of the circuit LCR as the output.

Report.—Plot the two efficiency curves, the first as a function of R, and the second, of E_B. Explain what is meant by neutralization. Would the amplifier function if L' were removed?

217. Modulation.—A continuous wave telegraph transmitter radiates a wave of constant amplitude and frequency, which produces in a distant receiving antenna tuned to it, an alternating current of the same frequency and an amplitude greatly reduced in magnitude. This received current, when rectified by appropriate means and passed through a telephone receiver, produces a steady displacement of the diaphragm. Under these circumstances the phone emits no sound. It merely clicks when the signal is turned on and off.

If, however, the amplitude of the emitted wave could be made to fluctuate by a microphone control in a manner corresponding to the frequency of the voice, then the rectified current through the phones of the receiving set would vary in a manner similar to the microphone current at the transmitter, and the diaphragm of the receiving phones would emit voice-frequency sound waves. The process of causing the amplitude of the radiated wave to vary in any required manner is called "Modulation." A radio telephone transmitter accordingly consists of an arrangement for producing continuous radio frequency currents coupled to an audio-frequency amplifier in such a manner that the radio-frequency output is controlled by sound waves striking the diaphragm of a microphone. It is standard practice to use at the input end of the radio-frequency equipment a master oscillator of low power whose frequency is stabilized by a quartz piezo-electric plate. The output of this oscillator is built up by a succession of radio-frequency amplifiers until the power of the final stage is that desired for the set. Modulation may be effected at any point in the radio-frequency train, but is usually applied at the last or next to the last stage.

218. Methods of Modulation.—The output of any radio-frequency oscillator or amplifier may be controlled by varying the voltage on either the grid or the plate of the tube and two methods of modulation have come into use which are known respectively as "Grid Potential" and "Plate Potential" modulation. The former requires less equipment but involves much more critical adjustment and is seldom used except in low-power portable sets where quality of reproduction is not of prime importance. Plate-potential modulation is much more stable, requires less critical adjustment of the equipment and is quite generally used by broadcast transmitters. It necessitates, however,

that the audio-frequency power be brought up to a higher level than is required in the case of grid-potential modulation.

Plate-potential modulation is based on the fact that, in any oscillator or amplifier with *tuned-circuit* coupling, the oscillatory current output is closely proportional to the voltage of the plate. In the case of the amplifier, ample grid excitation must be provided if this proportionality is to be realized. Two methods of effecting plate-potential modulation are in common use which are known respectively as the "constant current" and the "transformer" method. The constant current method is illustrated in Fig. 167 where a conventional oscillating circuit is shown at the left driven by the tube O, while the tube M is the modulator. Both tubes are supplied with power from the battery B through the choke coil L_2 of large inductance, e.g., 30

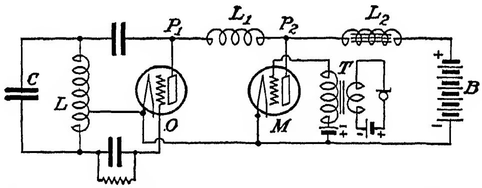

FIG. 167.—Constant current method of modulation.

henries. L_1 is a radio-frequency choke so designed that it offers a large impedance to radio-frequency currents and a small impedance to currents of audio frequency. It thus prevents radio-frequency voltage changes at P_1 from reaching P_2. The tube M takes no part in supplying radio-frequency power to the circuit LC. The grid bias of the modulator tube is so adjusted that it carries a plate current of the same order of magnitude as O when in oscillation. Suppose the grid of M is suddenly made more negative. Its plate current is decreased as is also the current furnished by B though the change in the latter is much smaller. This reduction in current through L_2 results in a self-induced voltage tending to maintain the current through it constant. The result is an increased voltage on the plates of both tubes since the induced voltage in L_2 is in the same direction as the battery voltage. The oscillating current in LC is accordingly increased.

When the voltage on the grid of M is made more positive, its plate current is increased, resulting in a slight increase in the

current from B and an induced voltage in L_2 opposing the battery voltage. This reduces the voltage on the plates of both tubes and consequently decreases the oscillating current. If the grid of the modulator tube is supplied with voice frequency voltages, the amplitude of the oscillating current in LC has voice-frequency variations.

The constant current method of modulation is subject to one serious objection; i.e., it is impossible to secure a high degree of modulation without the use of a large number of modulator tubes operated in parallel. To take full advantage of the available power of the transmitter, it is necessary that, by modulation,

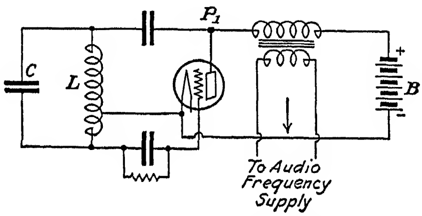

FIG. 168.—Transformer method of modulation.

the amplitude of the radio-frequency current should vary from zero to twice its unmodulated value, which requires that the plate current to the oscillator tube should also vary from zero to twice its unmodulated value. A consideration of Fig. 167 shows that, during modulation, the current to the plate of the oscillator tube increases by the same amount, neglecting the small variations in the current in L_2, that the plate current of the modulator tube decreases, and vice versa. The modulator, with the available energy stored in L_2, may be regarded as a generator delivering audio frequency power to the oscillator, and the magnitude of the modulation current to the oscillator tube is given by

$$I_m = \frac{E_p}{R_0 + R_m} \tag{16}$$

where E_p is the change of voltage produced at P_2 by modulation and R_0 and R_m are the plate-circuit resistances of the oscillator and modulator tubes respectively. The unmodulated plate current to the oscillator is

$$I_0 = \frac{E_B}{R_0} \tag{17}$$

where E_B is the voltage of the battery B. Assuming that E_p is given its maximum attainable value, E_B, then the ratio of I_m to I_0 is

$$\frac{I_m}{I_0} = \frac{R_0}{R_0 + R_m} \tag{18}$$

Thus it is seen that complete modulation can be secured by this method only when $R_m = O$, a condition which cannot be realized. By connecting several modulator tubes in parallel, R_m may be made considerably smaller than R_0. Five modulator tubes to three oscillators is common practice. One way of escaping this limitation is to use a higher plate voltage on the modulator tube than on the oscillator. With this arrangement, E_p may be made as large as desired and can be supplied to P_1 by a coupling condenser.

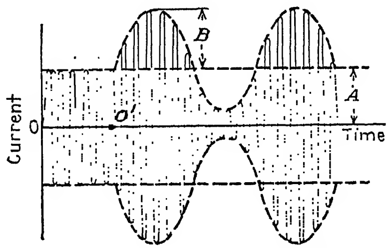

FIG. 169.—A Modulated wave.

The transformer method of modulation which is shown in Fig. 168 avoids the difficulty above discussed. It consists in replacing the modulator tube and reactance L_2 by the secondary of a transformer, the primary of which is supplied with audio-frequency power from an amplifier. By suitable choice of transformer ratio and adjustment of primary voltage, the modulation voltage at P_1 may be made any value desired and complete modulation easily be effected.

219. Mathematical Description of a Modulated Wave.—In Fig. 169 is shown a diagram of the current in the circuit LC of Fig. 167 when this current is modulated by a simple sine wave of voltage. From 0 to 0′, the current is unmodulated and may be represented by

$$i = A \sin \Omega t \tag{19}$$

where A is the amplitude and Ω the radian frequency of the radio frequency current. At 0′ modulation begins, and the amplitude of the oscillations now follows the law

$$\text{Amplitude} = A + B \sin \omega t \tag{20}$$

where B is the amplitude of modulation and ω the radian frequency of modulation. Replacing the constant amplitude A of equation 19 by its value in equation 20, we have

$$i = (A + B \sin \omega t) \sin \Omega t \tag{21}$$

By a simple trigonometric substitution, this becomes

$$i = A \sin \Omega t + \frac{B}{2} \sin (\Omega + \omega)t + \frac{B}{2} \sin (\Omega - \omega)t \tag{22}$$

This indicates that the current of Fig. 169 of supposedly constant radio frequency with variable amplitude, in reality consists of three distinct frequencies each of constant amplitude. The first term of equation 22 is the original current unmodulated; the second, a current of half the modulation amplitude with a frequency equal to the sum of the radio and modulation frequencies; and the third, one of the same amplitude as the second but with a frequency equal to the difference of the radio and modulation frequencies. That these three frequencies are actually present may be proven by testing with a sharply tuned wave meter.

The original unmodulated frequency is called the "carrier frequency" and the other two, "side frequencies." When modulating by voice waves, the modulation frequencies vary from perhaps 20 to 10,000 cycles per second and the side frequencies are no longer constant but are spread out into bands on each side of the carrier frequency and are spoken of as the "upper" and "lower" side bands.

220. Effective Value of a Modulated Current.—The effective value I_e of a modulated current is the equivalent direct current which liberates in a resistance R, the same amount of heat in a given time. Thus from equation 21,

$$\begin{aligned} I_e^2 RT &= R \int_0^T (A + B \sin \omega t)^2 \sin^2 \Omega t dt \\ &= R \int_0^T [A^2 \sin^2 \Omega t + 2AB \sin \omega t \sin^2 \Omega t + B^2 \sin^2 \omega t \sin^2 \Omega t] dt \end{aligned} \tag{23}$$

where T is the time of one complete period of the modulation frequency. Carrying out these integrations separately,

$$\int_0^T \sin^2 \Omega t dt = \int_0^T \frac{1 - \cos 2\Omega t}{2} dt = \frac{T}{2}$$

$$\int_0^T \sin \omega t \sin^2 \Omega t dt = \int_0^T \sin \omega t \left(\frac{1 - \cos 2\Omega t}{2}\right) dt = 0$$

$$\int_0^T \sin^2 \omega t \sin^2 \Omega t dt = \int_0^T \left(\frac{1 - \cos 2\omega t}{2}\right)\left(\frac{1 - \cos 2\Omega t}{2}\right) dt = \frac{T}{4}$$

Substituting in equation 23 and dividing out R and T

$$I_e^2 = \frac{A^2}{2} + \frac{B^2}{4} \tag{24}$$

In equation 24, $\frac{A^2}{2}$ is the square of the effective value of the unmodulated carrier current I_0. Putting $\frac{B}{A} = k$, the fraction of modulation, equation 24 becomes

$$I_e^2 = I_0^2\left(1 + \frac{k^2}{2}\right)$$

or

$$I_e = I_0\sqrt{1 + \frac{k^2}{2}} \tag{25}$$

Thus the effective value of an oscillatory current is increased by modulation. The amount of increase is independent of the frequency of either the oscillatory current or the modulation frequency, and depends only upon the fraction of modulation. A qualitative test for the proper functioning of any transmitter is to note whether or not the antenna current increases when modulating potentials are applied. Moreover, the fraction of modulation can easily be computed from equation 25 by measuring the modulated and unmodulated values of the current. Solving for k, we have

$$k = \sqrt{2}\sqrt{\left(\frac{I_e}{I_0}\right)^2 - 1} \tag{26}$$

where I_e and I_0 are the effective values of the respective currents as measured by a hot wire or a thermo-ammeter. When multiplied by 100 per cent equation 26 gives the so-called "per cent modulation."

221. Experiment 51. *Modulation by Transformer Method.*—Connect the apparatus as shown in Fig. 170 where 0 is a conven-

tional type oscillator supplied with power from the D.C. source E_B. T.A. is a thermo-ammeter for measuring its output current, while T is a modulation transformer supplied from an A.C. source, preferably 500 or 1,000 cycles per second. R is a simple receiving set consisting of a small coil L and tuning condenser placed at some distance from the oscillator with its coil so oriented that it picks up just enough energy from the oscillator to give a good signal in the phones when the D.P.D.T. switch S_1 is thrown downward. R_1 is a small variable resistance of such a magnitude that, with the switch thrown up, a good signal is heard. The purpose of the receiver and phone is to judge the quality of modulation by making a comparison between the audio-frequency output of the improvised transmitter and the audio-frequency input.

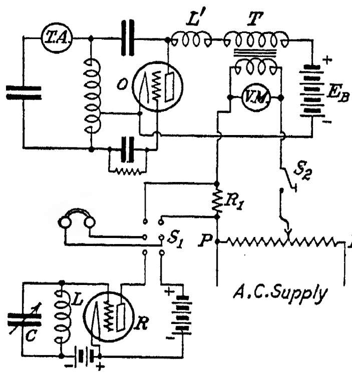

FIG. 170.—Transformer modulated oscillator.

Adjust the oscillator so that it is functioning normally without a modulating voltage applied. Determine the voltage ratio of the transformer T. Set the potential divider P.D. so as to give a small voltage on the transformer primary, close the switch S_2 and note the increase in the reading of T.A. Compute the peak value of the modulation voltage applied to the plate by multiplying the reading of the voltmeter by the transformer ratio times $\sqrt{2}$. This voltage divided by E_B, multiplied by 100, gives the percentage of modulation. Check this value by computing k from equation 26. Repeat the process for a series of modulation voltages up to and somewhat above 100 per cent, checking quality each time and noting at what per cent modulation distortion in the output begins to be noticeable.

Report.—Plot computed percentage of modulation, that is, peak value of voltage is secondary of the transformer divided by E_B, against measured percentages computed from equation 26. Explain what is meant by (*a*) modulation, (*b*) side bands. Why does the current in the oscillatory circuit increase with percentage modulation?

222. The Vacuum-tube Voltmeter.—From the standpoint of laboratory measurements, one of the most important applications of the three-element electron tube is its use as an alternating-current voltmeter. Due to the high impedance between grid and filament, the device consumes but a fraction of a microwatt of power when potentials within its range of applicability, i.e., a few hundredths to several volts, are impressed. Moreover, because of its quickness of response, it may be operated at very high as well as low frequencies, and, when proper precautions are taken, be calibrated at low frequencies, 60 cycles, and used at radio frequencies.

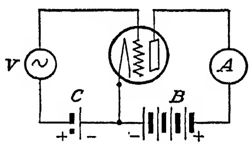

Fig. 171.—Simple vacuum-tube voltmeter circuit.

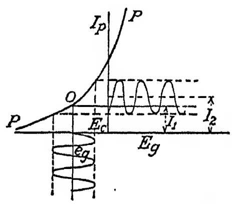

Fig. 172.—Operation of vacuum-tube voltmeter.

Several different circuits have been used each having its particular merits and limitations. One of the simplest is that shown in Fig. 171, where B is a battery of perhaps 90 volts; A, a milliammeter; C, a biasing battery; and V, the alternating potential to be measured.

The operation of the instrument is illustrated in Fig. 172 where the curve PP is the static characteristic of the tube for the particular plate potential E_B applied at B. The grid bias E_c is so chosen as to place the operating point at O on the curve; that is, when the grid and filament are short-circuited the steady plate current is I_1. If an alternating E.M.F. represented by the sine wave drawn downward in the figure, is applied to the grid, the current in the plate circuit changes as shown by the horizontally drawn wave. This latter, however, is not a sine wave because of the rectifying properties of the tube. Since the static characteristic has an upward curvature, a given positive change in grid potential causes a larger increment in plate current than a like change in the negative direction. The average value of the plate current, I_2, as read by a D.C. milliammeter, is greater than before. Thus an alternating voltage applied to the grid increases the

reading of the meter A, and when the indications of this D.C. meter corresponding to a series of A.C. voltages of known effective value have been determined, the device becomes a calibrated A.C. voltmeter.

The fact that the indications of the vacuum-tube voltmeter are independent of the frequency through wide frequency limits has already been mentioned. Another requirement of prime importance is that its indications should be independent of wave form. This means that all alternating voltages having the same effective value, but different wave forms, should produce the same response in the plate circuit milliammeter. Since the form of any wave is determined by the relative intensities of its various harmonics, it is necessary to investigate the effect upon the average value of the plate current of various combinations of frequencies simultaneously applied to the grid. In article 115 it was shown that the effective value of two alternating currents of different frequencies is the square root of the sum of the squares of their individual effective values. The reasoning there employed may be extended to any number of component currents, and to voltages as well. Thus, if E_1, E_2, E_3, . . . E_n are the effective values of n alternating E.M.F.'s impressed upon a circuit, their combined effective value E_e is

$$E_e = \sqrt{E_1^2 + E_2^2 + \cdots E_n^2} \tag{27}$$

To be sure that the vacuum-tube voltmeter is independent of wave form, it is necessary to determine under what conditions the change in the average value of the plate current is expressible as a function of the square root of the sum of the squares of the effective values of the component voltages impressed upon the grid.

For this purpose, let it be assumed that, for a fixed plate potential, the plate current may be expressed as a power series of the grid voltage, i.e.,

$$i_p = a_0 + a_1 e_g + a_2 e_g^2 + a_3 e_g^3 + a_4 e_g^4 + \cdots \tag{28}$$

For simplicity, assume that the impressed grid voltage consists of a fundamental frequency and a single harmonic of double that frequency with zero initial phase difference. That is,

$$e_g = E_1 \sin \omega t + E_2 \sin 2\omega t \tag{29}$$

The constant grid bias voltage is here omitted since its effect is merely to change the origin of the curve of equation 28, that is, to change the coefficients a_0 . . . a_n. The average value of the

plate current $\bar{i}_p$ after the alternating voltage is applied to the grid is

$$\bar{i}_p = a_0 + \frac{a_1}{T}\int_0^T e_g dt + \frac{a_2}{T}\int_0^T e_g^2 dt + \frac{a_3}{T}\int_0^T e_g^3 dt + \frac{a_4}{T}\int_0^T e_g^4 dt + \cdots \quad (30)$$

where e_g has the value indicated by equation 29. The work of carrying out these computations is somewhat laborious but involves merely the integrations between the limits O and T, of powers of the sine and cosine functions, and will not be given here. The result is

$$\bar{i}_p = a_0 + a_2\left(\frac{E_1^2}{2} + \frac{E_2^2}{2}\right) + 6a_4\left[\left(\frac{E_1^2}{2} + \frac{E_2^2}{2}\right)^2 + 2\left(\frac{E_1^2}{2}\cdot\frac{E_2^2}{2}\right)^2\right] + \cdots \quad (31)$$

From equation 27, the first bracketed term is the square of the effective value of the impressed grid voltage. Representing this by E_{ge}, remembering that a_0 is the plate current i_0 when the A.C. grid voltage is zero, and putting $\Delta i_p = \bar{i}_p - i_0$, we have

$$\Delta i_p = a_2 E^2_{ge} + 6a_4\left[E^4_{ge} + 2\left(\frac{E_1^2}{2}\frac{E_2^2}{2}\right)^2\right] + \cdots \quad (32)$$

Thus under the conditions here assumed, the plate current is not a function of the effective value of the grid voltage alone, but depends upon the relative values of the component voltages; that is to say, upon the form of the impressed wave. If, however, the second term of the right-hand member of equation 32 were zero, i.e., $a_4 = 0$, the change in plate current would be proportional to the square of the effective value of the grid voltage. But a_4 is the coefficient of the fourth-degree term in equation 28. Accordingly, if the tube is such that its static characteristic, throughout the range over which it is to be used, can be represented by a quadratic or a cubic function of the grid voltage, then indications of the vacuum-tube voltmeter are independent of wave form. A simple test of this property is to plot Δi_p against E_g^2 after making the calibration. If a straight line results, the calibration is independent of wave form.

223. Experiment 52. *Calibration of a Vacuum-tube Voltmeter.*—Connect the apparatus as shown in Fig. 173 where T is a vacuum tube and M.A., a milliammeter of such a range as to give nearly full-scale deflection for the saturation value of the plate current of the particular tube used. A.C. is a source of

alternating current which may conveniently be the secondary of a transformer giving 18 or 20 volts from a 60 cycle 110 volt power supply. R_1 and R_2 constitute a potential divider for impressing upon the grid any desired fraction of the secondary voltage as measured by the A.C. voltmeter V.M. B is an ordinary battery of about 90 volts, and C, a grid biasing battery with single cell pick off.

Make certain the tube is functioning properly and by adjusting the grid bias, find the maximum plate current with the particular

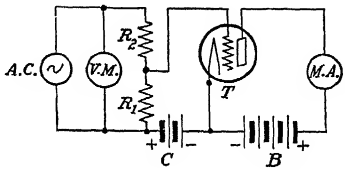

FIG. 173.—Circuit for calibrating a vacuum-tube voltmeter.

plate voltage used. Set the grid bias at such a value as to reduce the plate current to about one-tenth its maximum value. This point on the milliammeter scale then becomes the "false zero" for the calibration curve. Apply now the A.C. voltage and choose such values of R_1 and R_2 as to give eight or ten points on the milliammeter uniformly distributed throughout the plate current range. Use large values of R_1 and R_2 otherwise their power consumption may be so large as to injure them. A safe rule is that they should not consume more than 4 watts, computed from the square of the voltmeter reading divided by their series resistance.

Report.—Give the theory of the vacuum-tube voltmeter explaining its action in terms of the static characteristic of the tube. Plot, as a calibration curve, impressed A.C. volts as a function of plate current. What is meant by the "root mean square value" of an alternating current or voltage? What is meant by wave form and upon what does it depend? Prove that in a complex wave the effective value is the square root of the sum of the squares of the effective values of the components. Plot the square of the impressed voltages against plate current and draw a conclusion regarding the wave-form error in the particular vacuum-tube voltmeter you have calibrated.

224. The vacuum-tube oscillator lends itself very satisfactorily to the measurement of small capacitances. In Fig. 173*b*, a diagram is given showing how this may be accomplished. C_s is a variable condenser of known capacitance. This is adjusted for its maximum value (C_{s_1}). The wave meter (WM) is coupled

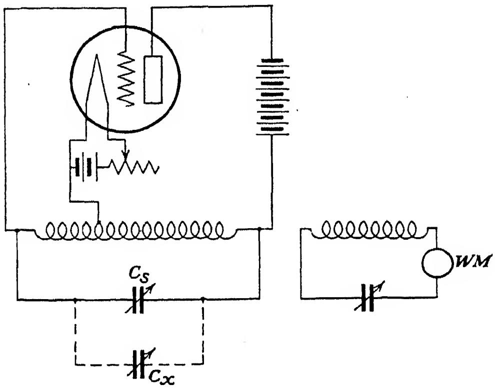

FIG. 137*b*.

inductively to the oscillator and is adjusted to resonance. If the capacitance to be measured (C_x) is now placed in parallel with C_s, the frequency of the oscillator is changed and the wave meter will not show resonance. It is then necessary to decrease C_s in order to cause the two circuits to be in tune. Let the value of C_s after the decrease be C_{s_2}. Then

$$C_x = C_{s_1} - C_{s_2}$$

225. Experiment 53.—*Measurement of Capacitance Using a Vacuum-tube Oscillator.*—Connect the apparatus as shown in Fig. 173*b*. Couple the wave meter as loosely as possible to still get a good indication of resonance. Calibrate a variable air condenser.

Report.—Discuss the theory of the Hartley oscillator. Why must the wave meter be coupled loosely?

If very small capacitances are to be measured the leads used in connecting C_s to C_x should be connected in the circuit when the initial adjustment of the wave meter is made. Why?

Plot a calibration curve for the condenser whose capacitance you measured.

226. Optional Set-up for Experiment 53.—If it is desired, the unknown condenser and C_s may be used in the wave meter circuit. In this case a variable or a fixed condenser of proper value may be used in the oscillating circuit. The procedure for calibrating will be the same as in the experiment already described.

APPENDIX

CALCULATION OF INDUCTANCE AND CAPACITANCE

In designing electrical apparatus and in checking the results of bridge measurements it is often advantageous to determine the inductance of coils by calculation from their dimensions and number of turns. In connection with its work in establishing primary units of inductance, the United States Bureau of Standards made an exhaustive study of the formulas for this purpose, and, besides extending those available at the time, developed a number of new ones. A comprehensive collection of inductance formulas, together with numerical examples, is given in the *Bulletin* of the Bureau of Standards, vol. 8, 1912, pp. 1 to 237. This publication is known also as Scientific Paper **169.** In another publication, "Radio Instruments and Measurements," Circular **74,** there is given a series of simplified formulas which yield results accurate to one-tenth of one per cent.

Several formulas, taken from Circular **74,** are given below. They apply to the coils most commonly used in every day practice. Lengths and other dimensions are expressed in centimeters, and the inductance calculated is given in microhenries. One henry = 10^3 millihenries = 10^6 microhenries = 10^9 C.G.S. electromagnetic units. It is assumed that the coil is placed in air or other medium whose permeability is unity, and that no iron is in the vicinity.

I. Single Layer Coil or Solenoid.—Nagaoka's Formula.

$$L = \frac{0.03948a^2n^2}{b}K \text{ microhenries} \tag{1}$$

where n = number of turns of coil

a = radius of coil, i.e., axis to center of any wire

b = length of coil, i.e., number of turns times distance between centers of adjacent turns.

K is a correction factor made necessary by the demagnetizing action of the ends of the coil and is a function of $\frac{2a}{b}$. Its value

may be read from Table I. If the coil is very long compared to its diameter, $K = 1$. Formula (1) takes no account of the size or shape of the cross-section of the wire and assumes that the

TABLE I.—VALUES OF K FOR USE IN FORMULA (1)

Diameter / Length	K	Difference	Diameter / Length	K	Difference	Diameter / Length	K	Difference
0.00	1.0000	−0.0209	2.00	0.5255	−0.0118	7.00	0.2584	−0.0047
.05	.9791	203	2.10	.5137	112	7.20	.2537	45
.10	.9588	197	2.20	.5025	107	7.40	.2491	43
.15	.9391	190	2.30	.4918	102	7.60	.2448	42
.20	.9201	185	2.40	.4816	97	7.80	.2406	40
0.25	0.9016	−0.0178	2.50	0.4719	−0.0093	8.00	0.2366	−0.0094
.30	.8838	173	2.60	.4626	89	8.50	.2272	86
.35	.8665	167	2.70	.4537	85	9.00	.2185	79
.40	.8499	162	2.80	.4452	82	9.50	.2106	73
.45	.8337	156	2.90	.4370	78	10.00	.2033	
0.50	0.8181	−0.0150	3.00	0.4292	−0.0075	10.0	0.2033	−0.0133
.55	.8031	146	3.10	.4217	72	11.0	.1903	113
.60	.7885	140	3.20	.4145	70	12.0	.1790	98
.65	.7745	136	3.30	.4075	67	13.0	.1692	87
.70	.7609	131	3.40	.4008	64	14.0	.1605	78
0.75	0.7478	−0.0127	3.50	0.3944	−0.0062	15.0	0.1527	−0.0070
.80	.7351	123	3.60	.3882	60	16.0	.1457	63
.85	.7228	118	3.70	.3822	58	17.0	.1394	58
.90	.7110	115	3.80	.3764	56	18.0	.1336	52
.95	.6995	111	3.90	.3708	54	19.0	.1284	48
1.00	0.6884	−0.0107	4.00	0.3654	−0.0052	20.0	0.1236	−0.0085
1.05	.6777	104	4.10	.3602	51	22.0	.1151	73
1.10	.6673	100	4.20	.3551	49	24.0	.1078	63
1.15	.6573	98	4.30	.3502	47	26.0	.1015	56
1.20	.6475	94	4.40	.3455	46	28.0	.0959	49
1.25	0.6381	−0.0091	4.50	0.3409	−0.0045	30.0	0.0910	−0.0102
1.30	.6290	89	4.60	.3364	43	35.0	.0808	80
1.35	.6201	86	4.70	.3321	42	40.0	.0728	64
1.40	.6115	84	4.80	.3279	41	45.0	.0664	53
1.45	.6031	81	4.90	.3238	40	50.0	.0611	43
1.50	0.5950	−0.0079	5.00	0.3198	−0.0076	60.0	0.0528	−0.0061
1.55	.5871	76	5.20	.3122	72	70.0	.0467	48
1.60	.5795	74	5.40	.3050	69	80.0	.0419	38
1.65	.5721	72	5.60	.2981	65	90.0	.0381	31
1.70	.5649	70	5.80	.2916	62	100.0	.0350	
1.75	0.5579	−0.0068	6.00	0.2854	−0.0059			
1.80	.5511	67	6.20	.2795	56			
1.85	.5444	65	6.40	.2739	54			
1.90	.5370	63	6.60	.2685	52			
1.95	.5316	61	6.80	.2633	49			

diameter of the wire is small compared to the dimensions of the coil, and that the coil is compactly wound.

II. Multiple Layer Coil.—For a long coil with few layers, the inductance is given by

$$L = L_S - \frac{0.01257n^2ac}{b}(0.693 + B_S) \text{ microhenries} \qquad (2)$$

where L_S = inductance of mean single layer given by formula (1)
n = number of turns of the coil
a = radius of coil measured from the axis to the center of cross-section of the winding
b = length of coil = distance between centers of turns times number of turns in one layer
c = radial depth of winding = distance between centers of two adjacent layers times the number of layers.
B_S = correction given in Table II in terms of the ratio $\frac{b}{c}$.

Table II.—Values of B_S for Use in Formula (2)

b/c	B_S	b/c	B_S	b/c	B_S
1	0.0000	11	0.2844	21	0.3116
2	0.1202	12	0.2888	22	0.3131
3	0.1753	13	0.2927	23	0.3145
4	0.2076	14	0.2961	24	0.3157
5	0.2292	15	0.2991	25	0.3169
6	0.2446	16	0.3017	26	0.3180
7	0.2563	17	0.3041	27	0.3190
8	0.2656	18	0.3062	28	0.3200
9	0.2730	19	0.3082	29	0.3209
10	0.2792	20	0.3099	30	0.3218

III. Short Circular Coil with Rectangular Cross Section.—For a coil having a shape such as shown in Fig. 174, the inductance is given by a formula due to Stefan. It is deduced on the assumption that the wire is rectangular in cross-section, and that the insulating space between turns is negligible. Further, the axial and radial dimensions of the winding are supposed to be small compared to the mean radius of the coil.

Let a = the mean radius of the winding measured from the axis to the center of the cross-section

b = the axial dimension of the cross-section

c = the radial dimension of the cross-section

d = $\sqrt{b^2 + c^2}$ = the diagonal of the cross-section

n = number of turns of rectangular wire.

There are two cases depending upon the relative values of b and c.

Case 1. $b > c$.

$$L = 0.01257an^2\left[2.303\left(1 + \frac{b^2}{32a^2} + \frac{c^2}{96a^2}\right)\log_{10}\frac{8a}{d} - y_1 + \frac{b^2}{16a^2}y_2\right] \qquad (3)$$

Case 2. $b < c$.

$$L = 0.01257an^2\left[2.303\left(1 + \frac{b^2}{32a^2} + \frac{c^2}{96a^2}\right)\log_{10}\frac{8a}{d} - y_1 + \frac{}{16a^{2}}\right. \qquad (4)$$

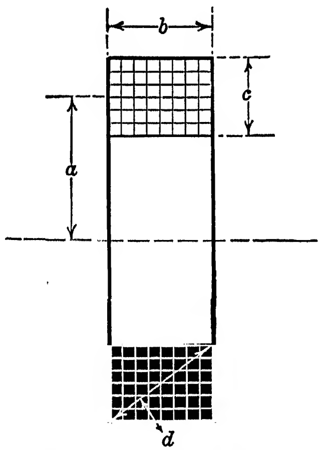

FIG. 174.—Multiple layer coil with winding of rectangular cross section.

The constants y_1, y_2, and y_3 depend upon relative values of b and c, and are given in Table III. The ratio of these quantities is always to be taken so as to give a proper fraction; i.e., in formula (3), use c/b, and in formula (4), use b/c. In eq. (3), y_1 is the same function of c/b that it is of b/c in eq. (4).

TABLE III.—CONSTANTS USED IN FORMULAS (3) and (4)

b/c or c/b	y_1	Difference	c/b	y_2	Difference	b/c	y_3	Difference
0	0.5000	0.0253	0	0.125	0.002	0	0.597	0.002
0.025	.5253	237						
.05	.5490	434	0.05	.127	5	0.05	.599	3
.10	.5924	386	.10	.132	10	.10	.602	6
0.15	0.6310	0.0342	0.15	0.142	0.013	0.15	0.608	0.007
.20	.6652	301	.20	.155	16	.20	.615	9
.25	.6953	266	.25	.171	20	.25	.624	9
.30	.7217	230	.30	.192	23	.30	.633	10
0.35	0.7447	0.0198	0.35	0.215	0.027	0.35	0.643	0.011
.40	.7645	171	.40	.242	31	.40	.654	11
.45	.7816	144	.45	.273	34	.45	.665	12
.50	.7960	121	.50	.307	37	.50	.677	13
0.55	0.8081	0.0101	0.55	0.344	0.040	0.55	0.690	0.012
.60	.8182	83	.60	.384	43	.60	.702	13
.65	.8265	66	.65	.427	47	.65	.715	14
.70	.8331	52	.70	.474	49	.70	.729	13
0.75	0.8383	0.0039	0.75	0.523	0.053	0.75	0.742	0.014
.80	.8422	29	.80	.576	56	.80	.756	15
.85	.8451	19	.85	.632	59	.85	.771	15
.90	.8470	10	.90	.690	62	.90	.786	15
0.95	0.8480	0.0003	0.95	0.752	0.064	0.95	0.801	0.015
1.00	.8483	......	1.00	.816	.....	1.00	.816	

IV. Coil of Round Wire Wound in a *Channel of Rectangular Cross-section.* If the insulation is not too thick, eqs. (3) and (4) give a very close approximation for the case in which ordinary round wire is used. When the percentage of the cross-section occupied by the insulating space is large, the following correction must be added to these formulas.

$$\Delta L = 0.01257an^2\left[2.303 \log_{10} \frac{D}{d_0} + 0.155\right] \qquad (5)$$

where D = distance between centers of adjacent wires
d_0 = diameter of the bare wire.

V. Inductance of a Flat Spiral.—Formula 4, which applies to the case $b < c$ may be extended to give the inductance of a flat spiral, where the following interpretations are given to the quantities a, b and c. Suppose the coil to be wound of flat ribbon with dimensions as given in Fig. 175. Let n be the number of turns whose width in the direction of the axis is w and whose thickness is t. The pitch D is the distance from the center of

one turn to the center of the next at the corresponding point. Then

$$b = w, \quad c = nD, \quad \text{and} \quad d = \sqrt{b^2 + c^2}$$

The mean radius of the equivalent coil is taken as

$$a = a_1 + \tfrac{1}{2}(n - 1)D$$

where a_1 is one half the distance AB of the figure, i.e., half the distance from the innermost end of the spiral across the center of the spiral to the opposite end of the innermost turn.

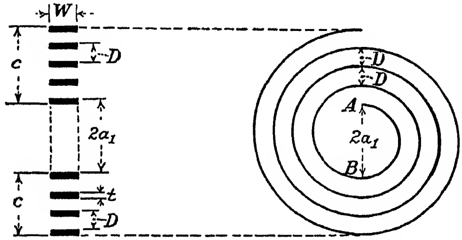

FIG. 175.—Sectional and side views of flat spiral.

If round wire is used, the inductance may be computed by the following formula in which a and c are determined as above, but b does not appear.

$$L = 0.01257an^2 \left\{ 2.303 \log_{10} \frac{8a}{c} - \frac{1}{2} + \frac{c^2}{96a^2} \left(2.303 \log_{10} \frac{8a}{c} + \frac{43}{12} \right) \right\} \quad (6)$$

VI. Self-inductance of Wires and Antennas.—For a single straight wire suspended at a considerable height above the ground, the inductance may be computed by the following formula

$$L = 0.002l\left[2.303 \log_{10} \frac{4l}{d} - 1 + \frac{\mu}{4}\right] \text{ micro-henries} \quad (7)$$

where l = length of wire
d = diameter of cross section
μ = permeability of the material of the wire

For all except iron wires, this becomes

$$L = 0.002\,l\left[2.303 \log_{10} \frac{4l}{d} - 0.75\right] \text{ micro-henries} \quad (8)$$

This assumes that the current in each unit of length is the same. When used as an antenna at a frequency well below its

natural frequency but high enough so that the current distribution throughout its length is non-uniform, the effective inductance of the antenna to be added to that of the loading coil is one-third the value computed by the above formula. See paragraph 190. For a cage antenna of small diameter with a liberal number of wires this formula gives a fair approximation.

CALCULATION OF CAPACITANCE

The following formulas[1] may be used to calculate the capacitance of condensers of the common forms. The dimensions of the condensers are measured in centimeters, and the capacitance is given in micro-microfarads. In these formulas, no correction is made for the curving of the electrostatic field at the edges of plates, etc., and it is assumed that the distance between plates is small compared to their linear dimensions.

VII. Parallel Plate Condenser

$$C = 0.0885K\frac{S}{T} \tag{9}$$

where S = surface area of one plate
T = thickness of dielectric
K = dielectric constant ($K = 1$ for air, and for most substances, lies between 1 and 10).

If, instead of a single pair of plates, there are N similar plates with dielectric between them alternate plates being connected in parallel,

$$C = 0.0885K\frac{(N-1)S}{T} \tag{10}$$

VIII. Variable Condenser with Semi-circular Plates

$$C = 0.1390K\frac{(N-1)(r_1^2 - r_2^2)}{T} \tag{11}$$

where N = total number of plates
r_1 = outside radius of the plates
r_2 = inside radius of the plates
T = thickness of dielectric
K = dielectric constant

This formula gives the maximum capacitance, i.e., when the movable plates are entirely within the spaces between the fixed plates. As the movable plates are rotated out, the capacitance decreases in direct proportion to the angle through which they are turned.

[1] *Cir.* **74**, U. S. Bureau of Standards, p. 235.

IX. Isolated Disk of Negligible Thickness

$$C = 0.354d \tag{12}$$

where d = diameter of the disk

X. Isolated Sphere

$$C = 0.556d \tag{13}$$

where d = diameter of the sphere

XI. Two Concentric Spheres

$$C = 1.112K \frac{r_1 r_2}{r_1 - r_2} \tag{14}$$

XII. Capacitance of Wires and Antennas.—1. *Single Long Wire Parallel to the Ground.*—For a single long wire of length l and diameter d suspended at a height h above the ground, the capacitance, depending upon the relative values of l and h, is given by one or the other of the following formulae:

For $\frac{4h}{l} \lesseqgtr 1$,

$$C = \frac{0.2416l}{\log_{10} \frac{4h}{d} - K_1} \text{ micro-microfarads} \tag{14}$$

For $\frac{4h}{l} \geqq 1$,

$$C = \frac{0.2416l}{\log_{10} \frac{2l}{d} - K_2} \text{ micro-microfarads} \tag{15}$$

The quantities K_1 and K_2 are defined as follows:

$$K_1 = \log_{10} \left\{ \frac{1 + \sqrt{1 + \left(\frac{4h}{l}\right)^2}}{2} \right\} \tag{16}$$

$$K_2 = \log_{10} \left\{ \frac{l}{4h} + \sqrt{1 + \left(\frac{l}{4h}\right)^2} \right\} \tag{17}$$

The magnitudes of these quantities for a series of values of $4h/l$ are given in the following table and intermediate values may be obtained by interpolation.

TABLE IV

$4h/l$	K_1	$l/4h$	K_2	$4h/l$	K_1	$l/4h$	K_2
0	0	0	0	0.6	0.035	0.6	0.247
0.1	0.001	0.1	0.043	.7	.045	.7	.283
.2	0.004	.2	.086	.8	.057	.8	.318
.3	0.009	.3	.128	.9	.069	.9	.351
.4	.016	.4	.169	1.0	.082	1.0	.383
.5	.025	.5	.209				

2. *Vertical Wire.*—Formula 15 holds accurately only when the height of an antenna is large compared to its horizontal flat top, and may then be used for the case in which the latter is zero, i.e., a single vertical wire. For this case, $K_2 = 0$. It also gives approximate values for a vertical wire whose lower end is connected to apparatus several meters above ground.

$$C = \frac{0.2416h}{\log_{10} \frac{2h}{d}} \tag{18}$$

XIII. Resonance Formulae for Oscillatory Circuit.—The following formulae are useful for computing the resonance wave length and frequency for an oscillatory circuit. Several combinations of the working units of inductance and capacitance are given with the corresponding constant. The following abbreviations are used:

λ = wave length in meters
n = frequency in cycles per second
$\omega = 2\pi n$ = frequency in radians per second

h = henries	f = farads
mh = milli-henries	μf = microfarads
μh = micro-henries	$\mu\mu f$ = micro-microfarads

$$\lambda = 1.884\sqrt{L(\mu h)\,.\,C(\mu\mu f)}$$
$$= 1{,}884\sqrt{L(\mu h)\,.\,C(\mu f)}$$
$$= 59{,}570\sqrt{L(mh)\,.\,C(\mu f)}$$
$$n = \frac{5033}{\sqrt{L(mh)\,.\,C(\mu f)}}$$
$$\frac{159{,}200}{\sqrt{L(\mu h)\,.\,C(\mu f)}}$$
$$\lambda = \frac{2.998 \times 10^8}{n} = \frac{2.998 \times 10^5}{\text{kilo-cycles}}$$

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